k-walks of Graphs

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ABSTRACT

We obtain various sufficient conditions for a graph to have a spanning closed walk meeting each vertex exactly k times or meeting each vertex at most k times. In particular, we generalise the result of Oberly and Sumner that every connected, locally connected $K_{1,3}$-free graph with at least three vertices is hamiltonian.

1. Introduction.

Our purpose is to generalise the concept of hamiltonicity by considering spanning closed walks in a graph which visit each vertex exactly k times, or at most k times. Jungreis [J] considered closed walks in a Cayley digraph of $\mathbb{Z}_m \otimes \mathbb{Z}_n$ visiting r vertices twice and the rest once. Broersma [B2] considered closed walks visiting each vertex of a graph exactly k times. We obtain sufficient conditions for the existence of such walks in several types of graphs.

All our graphs are simple, and we use the term *multigraph* at those times when multiple edges are permitted. We use G to denote an arbitrary graph. For an integer k, denote by $k \times G$ the multigraph obtained from G by multiplying all edges by k. An *exact k-walk* (or *k-walk*) of G is a connected spanning subgraph W of $(2k) \times G$, such that the degree of each vertex v in W is $2k$ (or is an even number which is at most $2k$, respectively). This nomenclature is motivated by the fact that Euler's Theorem implies that a k-walk possesses a closed walk traversing each edge exactly once (an Euler tour), and so a graph with a k-walk (or exact k-walk) possesses a closed walk passing through each vertex at most k times (or exactly k times, respectively). One interesting result from [B2, Corollary 3.3] is that if a graph has an exact k-walk then it has an exact $(k + 1)$-walk ($k \geq 1$).

Given two graphs G and H, the *composition* of G and H, denoted by $G[H]$, is defined as the graph with vertex set $V(G) \times V(H)$ and edge set $\{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)\}$. Note that for a graph with at least three vertices, every 1-walk is a Hamilton cycle. On the other hand, for $k \geq 2$, G has a k-walk (or exact k-walk) if and only if $G[K_k]$ (or $G[\overline{K}_k]$, respectively) has a Hamilton cycle. Thus we may use results on Hamilton cycles to obtain results on k-walks. There is a strong relationship between k-walks and the hamiltonicity of compositions, since if H has a Hamilton path then $G[H]$ is hamiltonian if and only if G has a $|V(H)|$-walk. We exploit a similar connection in examining the complexity of finding k-walks (see Section 6).

We use $\delta(G)$ (or $\Delta(G)$) to denote the minimum (or maximum, respectively) degree of a vertex in a graph G, and $\alpha(G)$ to denote the independence number of G. Also, G is $K_{1,k}$*-free* if no induced subgraph of G is isomorphic to $K_{1,k}$. Oberly and Sumner [OS] showed that every connected, locally connected $K_{1,3}$-free graph with at least three vertices is hamiltonian. Matthews and Sumner [MS] surveyed further results on $K_{1,3}$-free graphs and showed that any 2-connected $K_{1,3}$-free graph G with $\delta(G) \geq (|V(G)| - 2)/3$ has a Hamilton cycle. A classic result of Dirac [D] is that every graph G with $\delta(G) \geq |V(G)|/2$ and $|V(G)| \geq 3$ has a Hamilton cycle. Our main object is to give several related results for k-walks, as well as results relating to $\alpha(G)$, toughness, squares of graphs and planar graphs.

One of the devices used several times in our proofs is the consideration of an Euler tour T in a k-walk W. A vertex v of degree $2r$ in W must be met exactly r times by T, and so T can be partitioned into r subtours, say $T_1, \ldots, T_r$, each meeting v exactly once. We call these subtours the *branches* of T at v. For each vertex v of W choose an ordered labelling $T(v) = (v_1, \ldots, v_{2r})$ of the neighbours of v on T in the order in which they occur on T. Note that a neighbour of v on T may have several different labels. We shall write $v_i \sim v_j$ to mean that v_i and v_j are distinct labellings of the same vertex, and use vv_i to denote the unique edge of T from v to v_i.

We also use $N(v)$ to denote the set of neighbours of a vertex v in a graph G, and $NG(v)$ to denote the subgraph of G induced by $N(v)$. For a vertex v of a multigraph W, $d_W(v)$ denotes the degree of v in W, which is the number of edges incident with v.

2. Toughness and k-trees.

Let G be a graph and S a proper subset of $V(G)$. Let $c_0(G - S)$ denote the number of isolated vertices of $G - S$ and $c(G - S)$ the number of components of $G - S$. We first state a necessary condition for G to have a k-walk or an exact k-walk.

Lemma 2.1.
(i) If G has a k-walk then $c(G - S) \leq k|S|$ for all nonempty proper subsets S of $V(G)$.
(ii) If G has an exact k-walk then $c(G - S) + (k - 1)c_0(G - S) \leq k|S|$ for all nonempty proper subsets S of $V(G)$.

Proof.
(i) This follows since a k-walk of G must meet a vertex of S on passing between two components of $G - S$.
(ii) This is given in [B2, Proposition 2.1]. ∎

Following Chvátal [C], we say that G is *t-tough* for some $t > 0$ if G is connected and $c(G - S) \le |S|/t$ for all vertex cutsets S of G. Thus Lemma 2(i) can be restated as "if G has a k-walk then G is $(1/k)$-tough."

Remark 2.1. To see that the condition in Lemma 2(i) is not also sufficient, we create for any $\varepsilon > 0$ the following graph G which is $(1/k + 2/3k^2 - \varepsilon)$-tough and has no k-walk. We first define H to be the graph obtained from K_3 by attaching k pendant vertices at each of the three vertices. We then construct G from the disjoint union of $\overline{K}_s$ with $\lceil (sk+1)/2 \rceil$ copies of H by joining each vertex of $\overline{K}_s$ to every vertex in each copy of H. Given any $\varepsilon > 0$, we may choose s large enough so that G is $(1/k + 2/3k^2 - \varepsilon)$-tough. To see that G has no k-walk note that: any closed walk in G which meets each copy of H at least twice must meet some vertex of $\overline{K}_s$ at least $k + 1$ times, and, on the other hand, any spanning walk which meets some copy H_i of H exactly once, must meet some vertex of the K_3 contained in H_i at least $k + 1$ times.

Our next main object is to use a result of Sein Win to deduce that every $(1/(k-2))$-tough graph has a k-walk. A *k-tree* of a graph is a spanning tree with maximum degree k. We have the following relationship between k-trees and k-walks.

Lemma 2.2.
(i) If G contains a k-tree then G has a k-walk.
(ii) If G has a k-walk then G contains a $(k+1)$-tree.

Proof.
(i) Doubling the edges in a k-tree in G yields a k-walk of G.
(ii) Direct the edges of a k-walk of G to follow an Euler tour T. Delete from T any edge entering a vertex previously visited by the tour. The resulting multigraph, say H, is connected and has maximum degree at most $k + 1$. Any spanning tree of H is a $(k + 1)$-tree in G. ∎

Theorem 2.3. [SW] If G is connected, $k \ge 2$, and, for any subset S of $V(G)$, $c(G - S) \le (k-2)|S| + 2$, then G has a k-tree.

Corollary 2.4. If G is connected, $k \ge 2$, and, for any subset S of $V(G)$, $c(G - S) \le (k-2)|S| + 2$, then G has a k-walk.

We feel that Corollary 2.4 can probably be improved to the following.

Conjecture 2.1. If $k \ge 2$ then every $(1/(k-1))$-tough graph has a k-walk.

Remark 2.2. For the special case $k = 1$, Chvátal [C] has conjectured that there is some t for which every t-tough graph has a 1-walk. The lower bound 2 on such t was established by Enomoto et al. [EJKS], who constructed, for any $\varepsilon > 0$, a graph which is $(2 - \varepsilon)$-tough and has no 1-walk.

3. $K_{1,k+1}$-free graphs.

In this section we examine connected claw-free graphs in general, postponing extra connectivity considerations until the next section.

Theorem 3.1. Let G be a connected, $K_{1,k+1}$-free graph.
(i) G has a k-walk.
(ii) If $\delta(G) \geq k$ then G has an exact k-walk.

Proof. Let G be a connected graph. To prove (i), we show that for any connected graph G, there is a connected even spanning subgraph W of $m \times G$ for some m such that $d_W(v)$ is at most $2\alpha(NG(v))$ for all $v \in V(G)$. This suffices since $\alpha(NG(v)) \leq k$ for all v. Note firstly that G has a $\Delta(G)$-walk H, for example, $H = 2 \times G$. Let W be a $\Delta(G)$-walk of G for which $|E(W)|$ is minimised.

Suppose that there is some vertex v with $d_W(v) = 2r > 2\alpha(NG(v))$. Choose an Euler tour T in W, and let $y_1, \ldots, y_r$ denote edges incident with v in distinct branches of T at v. We complete the proof of (i) by finding a $\Delta(G)$-walk W' of G with $|E(W')| < |E(W)|$, yielding a contradiction. Observe that $W - \{y_i : i = 1, \ldots, r\}$ is connected. Hence, if y_i and y_j have the same end vertices for some $i \neq j$, then the deletion of y_i and y_j from W yields W' as required. Alternatively, $y_1, \ldots, y_r$ are incident with exactly $r > \alpha(NG(v))$ distinct vertices, say $u_1, \ldots, u_r$, in $N(v)$. Thus, $u_i u_j \in E(G)$ for some $i \neq j$. In this case, set $W' = W - \{y_i, y_j\} + u_i u_j$. This yields (i).

To prove (ii), we refine the proof of (i). We now assume $\delta(G) \geq k$. For a walk W, let $t(W)$ denote the number of edges of W which are members of multiple edges of cardinality at least 3. Let W be a $\Delta(G)$-walk of G with $\delta(W) \geq 2k$ for which $|E(W)|$ is minimised, and, subject to this, for which $t(W)$ is minimised. Suppose that $d_W(v) > 2k$ for some $v \in V(G)$.

A *triple* edge is a multiple edge of multiplicity exactly 3, and a *single* edge is an edge not in any multiple edge. We will find the following operations useful. Given a subgraph S of a subgraph U of $\Delta(G) \times G$, we define $U(S, a, b)$ to be the subgraph of $\Delta(G) \times G$ obtained from U by replacing every single edge of S by a multiple edge of cardinality a and every triple edge by a multiple edge of cardinality b. We define a subgraph R of W to be a *3,1-path* if it has distinct vertices $v = v_0, v_1, \ldots, v_q$ and edges $v_{2i+1}v_{2i+2}$, $0 \leq i \leq (q-2)/2$, which are single edges in R and W, and edges $v_{2i}v_{2i+1}$, $0 \leq i \leq (q-1)/2$, which are triple edges in R. Let $R = v_0, \ldots, v_q$ denote a maximal 3,1-path with $v = v_0$. We consider two cases. Note that the first includes $q = 0$.

Case 1. q is even.

First suppose that v_q is incident with no multiple edge of W of multiplicity at least 3. Put $W_1 = W(R, 3, 1)$, and note that $|E(W_1)| = |E(W)|$, $t(W_1) = t(W)$ and that $d_{W_1}(v_q) \geq 2k+2$, even if $q = 0$. Let s denote the number of vertices adjacent to v_q by single edges of W_1, and m the number of vertices adjacent to v_q by multiple edges of W_1. Denote these m vertices by $u_1, \ldots, u_m$. As in the proof of (i), choose an Euler tour T in W_1. Then at least $\lceil s/2 \rceil$ single edges incident with v_q are in distinct branches of T at v_q. Let y_i, $i = 1, \ldots, \lceil s/2 \rceil$, denote a set of such edges, and let u_{i+m} denote the other end vertex of y_i. Note that v_q is incident with at most one triple edge, and no edge of multiplicity greater than 3, in W_1, and so $m + \lceil s/2 \rceil \geq (d_{W_1}(v_q) - 1)/2 > k$. Hence, $u_i u_j \in E(G)$ for some i and j. Let x_1 denote y_{i-m} if $i > m$, and one of the edges $v_q u_i$ otherwise. Similarly, let x_2 denote y_{j-m} if $j > m$, and one of the edges $v_q u_j$ otherwise. Then $\{x_1, x_2\}$ is not a cutset of W_1, and so $W_2 = W_1 - \{x_1, x_2\} + u_i u_j$ is a $\Delta(G)$-walk of G with $\delta(W_2) \geq 2k$ and $|E(W_2)| = |E(W)| - 1$. This is a contradiction.

It follows that v_q is incident with a multiple edge of W of multiplicity at least 3. By the maximality of R, the multiple edge is incident with v_p for some $p < q - 1$. Then $R = R_1 \cup R_2$ where $R_1 \cap R_2 = \{v_p\}$, R_1 is the path-like subgraph of R between v_0 and v_p, and R_2 is the part between v_p and v_q. If $v_p = v_0$ then $R_1 = \{v_p\}$. Let R_3 be R_2 with a triple edge added between v_q and v_p. Put $W_1 = W(R_3, 2, 2)$. If p is odd, then $\delta(W_1) \geq 2k$, $|E(W_1)| = |E(W)|$ and $t(W_1) < t(W)$, contradicting the choice of W. Otherwise, put $W_2 = W_1(R_1, 3, 1)$ and a similar contradiction is reached. This finishes Case 1.

Case 2. q is odd.

First suppose that v_q is incident with no single edge of W. Then with $W_1 = W(R, 3, 1)$, we have $|E(W_1)| < |E(W)|$. This yields a contradiction unless $d_{W_1}(v_q) \leq 2k - 2$. Since all edges of W_1 incident with v_q are multiple, except perhaps $v_{q-1}v_q$, and $d_G(v_q) \geq k$, it follows that some edge $v_q u$ of G is not in W_1. Set $W_2 = W_1 + 2v_q u$. Then $|E(W_2)| = |E(W)|$ and $t(W_2) < t(W)$, a contradiction.

It follows that v_q is incident with a single edge, say x, of W. By the minimality of W, $x = v_q v_p$ for some $p < q - 1$. Then $R = R_1 \cup R_2$ where $R_1 \cap R_2 = \{v_p\}$, R_1 is the part of R between v_0 and v_p, and R_2 is the part between v_p and v_q. Let $R_3 = R_2 + v_p v_q$. The rest of the argument is as in Case 1, with the two subcases p even and p odd interchanged. ∎

Remark 3.1. Theorem 3.1 is sharp in the following sense. Since $K_{1,k}$ has no $(k-1)$-walk, (i) is not true with k replaced by $k - 1$. Similarly, for any $k \geq 2$, $K_{k,k-1}$

has no exact k-walk. Thus for any $k \geq 2$ there exists a connected $K_{1,k+1}$-free graph G with $\delta(G) = k - 1$ and no exact k-walk. Thus (ii) is false if $\delta(G) \geq k$ is replaced by $\delta(G) \geq k-1$.

4. Connectivity.

We next generalise the main theorem of [OS] by showing that the conclusion of Theorem 3.1 (i) can be strengthened if G is in addition locally connected; that is, $N(v)$ is connected for all $v \in V(G)$.

Theorem 4.1. For $k \geq 1$, every connected, locally connected $K_{1,k+2}$-free graph with at least two vertices has a k-walk.

Proof. Let G be a connected, locally connected $K_{1,k+2}$-free graph with at least two vertices. Then $\alpha(NG(v)) \leq k + 1$ for all $v \in V(G)$. By Theorem 3.1(i), G has a $(k + 1)$-walk, say W. Let $g(W)$ denote the number of vertices of degree $2k + 2$ in W, and choose W so that $g(W)$ is minimised. Assume that for some vertex v, $d_W(v) = 2k + 2$. We will show how to obtain a $(k + 1)$-walk W' of G which contradicts the minimality of W.

Let T be an Euler tour in W, and let S denote the set of edges in T incident with v. If $x \in T(v)$, we use x' to denote the element of $T(v)$ such that vx' is the other edge in S in the same branch of T as vx. As in the proof of Theorem 3.1(i) (but minimising $g(W)$ this time, rather than $|E(W)|$), if vx and vy are any two edges in S in distinct branches of T at v, then $x \neq y$ and $xy \notin E(G)$. Thus, if $vx_1, \ldots, vx_{k+1} \in S$ are in distinct branches of T at v, then $x_1, \ldots, x_{k+1}$ form an independent set. So since $\alpha(NG(v)) \leq k + 1$, we have that for each i, either $x_i \sim x_i'$ or $x_i x_i' \in E(G)$. Note that the branches of T at v can intersect only at v, since otherwise T can be re-routed so that the conditions above are not satisfied.

Let P be a shortest path in $NG(v)$ between vertices in distinct branches of T. The local connectivity ensures the existence of P. We can assume that W and v have been chosen so that the length of P is as small as possible (subject to the minimality of $g(W)$), and that subject to these conditions, $|E(W)|$ is minimised. By the previous paragraph, the length of P is at least 2. Also, if P has length at least 4, then a central vertex of P, together with $x_1, \ldots, x_{k+1}$, is an independent set in $NG(v)$, a contradiction. Thus, P has length at most 3. Without loss of generality, assume P is from x_1 to x_2, and let u denote the first vertex of P, apart from x_1, for which $d_W(u) \geq 2k$. If no such u exists, then we can obtain W' from W by replacing the edges vx_1 and vx_2 with the path P, to get $g(W') < g(W)$. Let $P(x_1, u)$ denote the set of edges of P from x_1 to u.

Let $w_1, \ldots, w_k$ be labelled vertices in $T(u)$ such that $uw_1, \ldots, uw_k$ are in distinct branches of T at u, where v is in the same branch at u as w_1, and let uw_{k+1} be another edge in that branch. By the minimality of $|E(W)|$, we can assume $w_1, \ldots, w_{k+1}$ are all distinct and independent, except perhaps for $w_1 \sim w_{k+1}$ or $w_1w_{k+1} \in E(G)$. But in either of these two cases we can modify W by deleting vx_1, uw_1 and uw_{k+1}, and inserting $P(x_1, u)$, the edge vu, and w_1w_{k+1} if $w_1 \sim w_{k+1}$ is false, to obtain a $(k+1)$-walk in which P is shorter or G is decreased, a contradiction. Hence $w_1 \sim w_{k+1}$ is false, and $w_1w_{k+1} \notin E(G)$.

It follows that every neighbour of u other than $w_1, \ldots, w_{k+1}$ is adjacent to at least one of the vertices $w_1, \ldots, w_{k+1}$; that is, to a neighbour of u on T. In particular, assume $vw_i \in E(G)$. If uw_i is in the same branch of T at v as x_j and x_j', where $j \neq 1$, we set $W' = W + \{vw_i, x_jx_j'\} + P(x_1, u) - \{vx_j, vx_j', uw_i, vx_1\}$, and remove the loop x_jx_j' if $x_j = x_j'$. This gives the desired walk W' with $g(W') < g(W)$. Hence, recalling that the branches at v are disjoint except at v, we see that u appears only in the same branch of T at v as x_1 and x_1'. Similarly, we find that if u' is the last vertex of P, apart from x_2, for which $d_W(u') \geq 2k$, then u' is in the same branch of T at v as x_2 and x_2'. Immediately, we obtain $u \neq u'$ and P has length 3. Thus, $uu' \in E(G)$. Hence, by the remark above, u' is adjacent to a neighbour of u on T, say w, and by symmetry, u is adjacent to a neighbour of u' on T, say w'. We can now set $W' = W + \{uw', u'w, x_1x_1'\} - \{vx_1, vx_1', uw, u'w'\}$, and remove x_1x_1' if it is a loop, to obtain the desired walk W' with $g(W') < g(W)$. ∎

We next examine global connectivity.

Theorem 4.2. If $j \geq 1$, $k \geq 3$ and G is j-connected and $K_{1,j(k-2)+1}$-free then G has a k-walk.

Proof. Let S be a proper subset of $V(G)$. Since G is j-connected, each component of $G - S$ is joined to at least j vertices in S, and since G is $K_{1,j(k-2)+1}$-free, each vertex in S is joined to at most $j(k-2)$ components of $G - S$. Hence, $c(G - S) \leq (k-2)|S|$. The theorem now follows from Corollary 2.4. ∎

Note that Theorem 3.1(i) is a strengthening of Theorem 4.2 with $j = 1$. Also, Theorem 4.2 improves Theorem 4.1 whenever $k \geq 6$ in Theorem 4.1 because all locally connected graphs other than K_2 are 2-connected. We believe that Theorem 4.2 can be sharpened as follows.

Conjecture 4.1. If $j \geq 1$, $k \geq 2$ and G is j-connected and $K_{1,jk+1}$-free then G has a k-walk.

Remark 4.1. The graph $K_{j,jk+1}$ has no k-walk. Hence, Conjecture 4.1 would be a best possible strengthening of Theorem 4.2 for $k \geq 2$. However, for $k = 1$, the graph obtained by expanding each vertex of the Petersen graph to a triangle is $K_{1,3}$-free and 3-connected and has no 1-walk, and the Meredith graphs [M] are r-connected, r-regular (and hence $K_{1,r+1}$-free) and have no 1-walk. A related conjecture in [MS] is that every $K_{1,3}$-free 4-connected graph has a 1-walk. We would like to ask how much this conjecture might be strengthened, as follows.

Question. If $j \geq 4$ and G is j-connected and $K_{1,j}$-free, does G have a 1-walk?

Theorems 4.1 and 4.2 also suggest the following.

Conjecture 4.2. If $j \geq 0$, $k \geq 1$ and G is connected, locally j-connected and $K_{1,(j+1)k+1}$-free then G has a k-walk.

Remark 4.2. Conjecture 4.2 is a common generalisation of Theorem 3.1(i) (when $j = 0$) and a conjecture of Oberly and Sumner [OS] (when $k = 1$). Since connected, locally j-connected graphs are $(j + 1)$-connected (except for K_2), Theorem 4.2 implies the weakened version of Conjecture 4.2 for $K_{1,(j+1)(k-2)+1}$-free graphs. If true, this conjecture is sharp, in view of the graph $K_{j+1} + \overline{K}_r$ obtained by joining each vertex of K_{j+1} to each vertex of $\overline{K}_r$, where $r = (j + 1)k + 1$.

It is possible that local connectivity conditions facilitate the appearance of k-trees. The truth of the following conjecture would go one step closer to establishing Conjecture 4.2, by Lemma 2.2(i).

Conjecture 4.3. If $j \geq 1$, $k \geq 2$ and G is connected, locally j-connected and $K_{1,(j+1)(k-1)+2}$-free then G has a k-tree.

Remark 4.3. If true, this conjecture is sharp, in view of $K_{j+1} + \overline{K}_r$. Any k-tree T in this graph requires at least $j + r$ edges. But every edge is incident with one of the vertices in K_{j+1}, and so T has at most $(j + 1)k$ edges. Hence, $r \leq (j + 1)(k - 1) + 1$.

5. Minimum degree, independence number, squares of graphs and planar graphs.

A *D_λ-cycle* in a graph G is a cycle C such that all components of $G - C$ have less than λ vertices. Clearly, $G[K_k]$ has a D_k-cycle if and only if G has a k-walk.

Theorem 5.1. If G is connected, $k \geq 2$ and $\delta(G) > (|V(G)| - 1) / (k + 1)$ then G has a k-walk.

Proof. We will use the following result implied by Veldman [V, part of Theorem 4]. Suppose $k \geq 2$ and G is a k-connected graph, and that the vertices of each connected subgraph of G with k vertices are adjacent to more than $(|V(G)| - 1) / (k + 1)$ other vertices. Then G has a D_k-cycle.

Consider $H = G[K_k]$. We shall refer to the K_k-subgraphs of H corresponding to vertices of G as *inflated vertices*. Noting that $|V(H)| = k|V(G)|$, that H is k-connected, and that each connected subgraph F of H with k vertices has more than $k(|V(G)| - 1) / (k + 1)$ neighbours in $V(H) \setminus V(F)$, we may apply Veldman's theorem to deduce that H has a D_k-cycle. ∎

Remark 5.1. If we require a minimum degree condition on G for an exact k-walk (rather than a k-walk as in Theorem 5.1) then the best we can do is $|V(G)| / 2$ for all k. The fact that all graphs G of minimum degree at least $|V(G)| / 2$ have a k-walk follows from Dirac's Theorem [D]. To see that we cannot do any better, consider $K_{m+1, m}$.

Recently Fraisse [F2, Corollary 1] showed that if G is a k-connected graph such that the degree sum of any $k + 1$ independent vertices is at least $|V(G)| + k(k - 1)$, then G has a D_k-cycle. Applying this result instead of [V, Theorem 4] in the proof of Theorem 5.1, we may deduce the stronger:

Theorem 5.2. If G is connected and every set of $k + 1$ independent vertices of G have degree sum at least $|V(G)|$ then G has a k-walk. ∎

It follows trivially from Theorem 3.1 that every connected graph G has an $\alpha(G)$-walk. This result may be extended for graphs of higher connectivity, as follows.

Theorem 5.3. Let G be a j-connected graph. Put $k = \lceil \alpha(G) / j \rceil$. Then G has a k-walk.

Proof. Again consider $H = G[K_k]$. Since H is kj-connected and $kj \geq \alpha(H) = \alpha(G)$, it follows from the Chvátal-Erdos Theorem [CE] that H is hamiltonian. ∎

Fleischner [F1] has shown that the square of a 2-connected graph has a 1-walk. Using this result we deduce the following.

Theorem 5.4. If G is connected then G^2 has a 2-walk.

Proof. Since $G[K_2]$ is 2-connected and $G^2[K_2] = G[K_2]^2$, it follows from [F1] that $G^2[K_2]$ is hamiltonian. ∎

If G has minimum degree 2 then Theorem 5.4 may be strengthened as in the next theorem. We first need a lemma for trees.

Lemma 5.5. If T is a tree then T^2 has a 2-walk W such that for all $v \in V(T)$, $d_W(v) = 2$ iff $d_T(v) = 1$.

Proof. Let $n = V(T)$ and let u be an arbitrary vertex of T which we will call a *root*. We strengthen the statement to be proved by asserting that, in addition to W, there is a 2-walk W' such that for all $v \in V(T)$, $d_{W'}(v) = 2$ iff $d_T(v) = 1$ or $v = u$. This is proved by induction on n. If $n = 2$ then it is immediate, so take $n \geq 3$. Let $T(u)$ denote the subtree of T induced by u and its neighbours. We can assume that for each component H of $T - u$, rooted at the neighbour of u in H, there is a 2-walk in H^2 of the type of W'. The union of these walks over all components H, together with a 1-walk in $T(u)^2$, yields the desired 2-walk W'. (Note that if any of the components is a single vertex, its 2-walk contains no edges.) Otherwise, we can assume that $d(u) \geq 2$, and then instead of a 1-walk in $T(u)^2$, use a 2-walk in which u is the only vertex of degree 4. This yields the walk W.

Theorem 5.6. If G is connected and $\delta(G) \geq 2$ then G^2 has an exact 2-walk.

Proof. Let T be a spanning tree of G and let H denote the subgraph of G induced by the endvertices of T. Let F be a spanning subgraph of H such that $d_F(v) \geq 1$ for all $v \in V(H)$ with $d_H(v) \geq 1$ and such that $|E(F)|$ is minimal. Clearly F is a spanning forest of H and each component of F is a star. Let S_i denote the set of vertices in F of degree i, and let M denote a set of edges of $G - T$ covering all the members of S_0, each edge containing one member of S_0. Define a spanning subgraph G' of G by $E(G') = E(T) \cup E(F) \cup M$. All vertices in S_0 and S_1 have degree 2 in G'. Let R denote a subset of $S_0 \cup S_1$ which contains all vertices but one in each component of F. (The only case in which there is some choice for membership in R is for those components of order 2.) Slicing each vertex of G' in R into two vertices of degree 1, we obtain a tree T' whose endvertices are the vertices coming from R. By Lemma 5.5, T'^2 has a 2-walk in which all these vertices have degree 2 and the rest have degree 4. This induces an exact 2-walk in G'^2 and hence in G^2. ∎

Tutte [T] has shown that every 4-connected planar graph is hamiltonian. On the other hand, $K_{2,2k+1}$ is an example of a 2-connected planar graph which has no k-walk for any $k \geq 1$. For the remaining case of 3-connected planar graphs, Barnette [B] has shown that all such graphs have a 3-tree. Using Lemma 2.2(i) we deduce the next result.

Theorem 5.7. Every 3-connected planar graph has a 3-walk. ∎

Perhaps the following stronger assertion is valid.

Conjecture 5.1. Every 3-connected planar graph has a 2-walk.

Note that if Conjecture 5.1 were true then, by Lemma 2.2(ii), it would generalise Barnette's result on 3-trees.

6. NP-completeness of k-walk problems.

It was shown in [B2] that the problem of whether a given graph has an exact k-walk is NP-complete. The proof was by transformation of an arbitrary graph G to a graph G' such that G has a Hamilton cycle iff G' has an exact k-walk. The NP-completeness of the exact k-walk problem thus follows from the NP-completeness of the Hamilton cycle question. In fact, with the proof given, G has a Hamilton cycle iff G' has any k-walk, and thus the question of whether a given graph has a k-walk is NP-complete. However, the graphs G' have many cut-vertices, and so it is natural to ask whether the restriction of the question to more highly connected graphs is still NP-complete. Using the conventions of Garey and Johnson [GJ], we may state the problems precisely as follows.

K-WALK IN J-CONNECTED GRAPH
Instance: j-connected graph G.
Question: Does G have a k-walk?

EXACT K-WALK IN J-CONNECTED GRAPH
Instance: j-connected graph G.
Question: Does G have an exact k-walk?

We generalise the result given in [B2] to the following.

Theorem 6.1. For k and j fixed, K-WALK IN J-CONNECTED GRAPH and EXACT K-WALK IN J-CONNECTED GRAPH are NP-complete.

Proof. We give a polynomial reduction from HAMILTON CYCLE to each problem. Let G be an arbitrary graph with $|V(G)| \geq 2$, and form the composition $H = G[K_j]$. To each inflated vertex of H (in the terminology of the proof of Theorem 5.1), join $jk - 1$ separate copies of K_j, to obtain G'. Then a k-walk in G' uses at most two edges of H incident with any inflated vertex, and so yields a 1-walk of G. The converse also holds. In addition, G' is j-connected. Thus, we have reduced HAMILTON CYCLE to K-WALK IN J-CONNECTED GRAPH. The proof for exact k-walks is exactly the same since if G has a 1-walk it follows that G' has an exact k-walk. ∎

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