BRACED EDGES IN PLANE TRIANGULATIONS

by

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ABSTRACT

A plane triangulation is an embedding of a maximal planar graph in the Euclidean plane. Foulds and Robinson (1979) first studied the problem of transforming one triangulation to another by a sequence of diagonal operations, where a diagonal operation deletes one edge and inserts the other diagonal of the resulting quadrilateral face. An edge which cannot be removed by a single diagonal operation is called braced. This paper is a study of the possible number and distribution of braced edges in a triangulation. It shows that at most $2n-4$ edges of a triangulation of order n can be braced, and that for any $r \leq 2n-4$ (with exactly one exception) there is a plane triangulation of order n with r braced edges, so long as n is large enough.

1. What are Braced Edges?

A *plane triangulation* T is an embedding of a maximal planar graph in the euclidean plane. The triangulation T is of *order* n if it has n *vertices*, and then Euler's polyhedral formula shows it has $3n-6$ *edges* and $2n-4$ *faces*, all triangles (that is, regions bounded by three vertices and three edges).

Suppose T has order $n \geq 4$. Then with each edge vw of T we can associate a pair of distinct vertices $\{x,y\}$, where each is the third vertex of a face incident with vw. If T contains an edge xy, we say that vw is *braced*, and xy is the *edge which braces* vw; if there is no edge incident with both x and y, we say that vw is *unbraced* (Figure 1). If

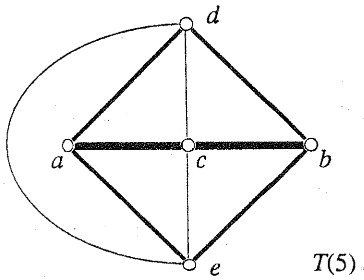

FIGURE 1. *A plane triangulation of order* 5. *The edges ac and bc are braced by de; ad and bd are braced by ce; ae and be are braced by cd; cd, ce and de are unbraced.*

an edge is unbraced, it can be deleted and the resulting quadrilateral face can have its other diagonal drawn in to produce a new plane triangulation. This is the *diagonal* *operation* studied in [1], and shown there to be essentially capable of transforming any plane triangulation of order n into any other. A braced edge may be regarded as an obstruction to diagonal operations, so it is of interest to study the possible number and distribution of braced edges in a plane triangulation.

As shown in [1], if T is any plane triangulation of order $n \geq 5$, any edge which braces another must itself be unbraced. (Here we shall refer to this result as *Theorem 0.*) Thus plane triangulations of order $n \geq 5$ always contain unbraced edges. But a single edge can brace more than one edge, so the possible proportion of braced to unbraced edges is not apparent. We call an unbraced edge *neutral* if it does not brace any other edge. The triangulation in Figure 1 has no neutral edges. The possible number and distribution of neutral edges are also matters of some interest.

In section 5 of the paper we show that any fixed number of braced edges can be achieved by triangulations of all sufficiently large orders. In section 7 we establish that the maximum number of braced edges in a triangulation of order n is $2n-4$ when $n \equiv 2 \pmod 3$ and $2n-5$ otherwise, and in Section 8 we describe configurations which achieve these maximum values. Finally, Theorem 4 of Section 9 summarizes our result that for any r less than the maximum (with exactly one exception) there is a triangulation with exactly r braced edges.

2. Triangulations of Small Order

Let us begin by examining the (equivalence classes of) plane triangulations of small order. This will lay the foundation for our subsequent results.

The plane triangulation $T(3)$ of order 3 corresponds to a plane embedding of the complete graph K_3. It is degenerate in that its two faces have the same boundary C, and our definition of braced edges does not apply. However, the plane triangulation $T(4)$ of order 4 can be regarded as a refinement of it, obtained by insertion of a vertex a in the interior of C, the resultant triangulation of the interior being unique. Hence $T(4)$ is the unique plane triangulation of order 4. It corresponds to a plane embedding of the complete graph K_4, and all 6 of its edges are braced. Continuing with the triangulation just obtained, we can further refine it by insertion of a vertex b in the exterior of C, the resultant triangulation of the exterior being unique. This produces a plane triangulation $T(5)$ of order 5 in which the 6 edges incident with a or b are all braced, while the 3 edges of C are unbraced but none is neutral (Figure 1). It is straightforward to verify that $T(5)$ is the unique plane triangulation of order 5. It is a plane embedding of K_5-e, the complete graph of order 5 with one edge deleted.

At order 6 we find two new phenomena appearing. There is more than one plane triangulation of order 6 – in fact, there are three. And there is more than one maximal planar graph of order 6 – in fact, there are two, obtained from K_6 by deleting either three independent edges or a path of three edges. The former corresponds to a unique plane triangulation $T(6,1)$ of order 6, with no braced edges (all edges are neutral). The latter corresponds to the other two plane triangulations of order 6, $T(6,2)$ and $T(6,3)$, each with 7 braced edges and no neutral edges (Figure 2). If regarded as sphere triangulations, $T(6,2)$ and $T(6,3)$ would be equivalent.

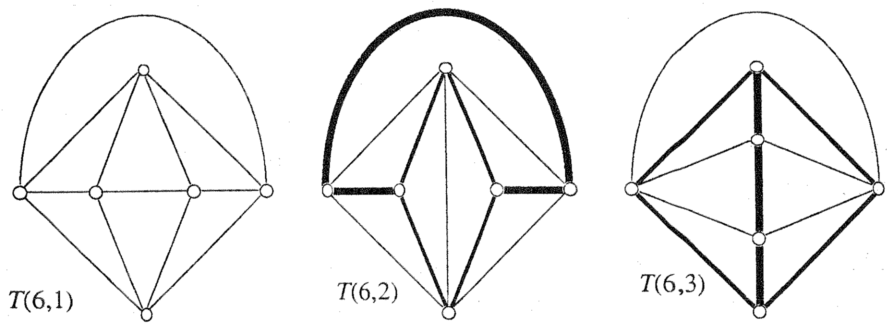

FIGURE 2. *The three plane triangulations of order* 6.

The plane triangulations $T(6,2)$ and $T(6,3)$ can be derived from $T(4)$ by refining (triangulating) two faces, either two interior faces or the exterior face and one interior face. Moreover, $T(6,1)$ cannot arise in this way since it has no vertex of degree 3.

3. Triangulating a Face

A plane triangulation T of order n can be refined into a plane triangulation T' of order $n+1$ by *triangulating a face*. If F is any face of T, the operation of triangulating F will be denoted by $Y(F)$. If the boundary of F is the cycle $\partial F := abca$, applying $Y(F)$ inserts a new vertex v and three new edges av, bv and cv into F (Figure 3). The new edges av, bv and cv are braced, by bc, ac and ab, respectively. Therefore in T', provided $n \geq 4$, the edges of the cycle $abca$ are unbraced (but not neutral), by Theorem 0, regardless of the status of those three edges in T.

We shall say that a face F of a plane triangulation is *braced* if it is incident with at least one braced edge, that is, if at least one edge in the boundary cycle ∂F is braced. Otherwise F is *unbraced*. Further, we shall describe a braced face as *singly, doubly* or *triply* braced, according to the number of braced edges in ∂F. With this terminology we can say that triangulating a face of a plane triangulation of order $n \geq 4$ replaces it with three doubly braced faces.

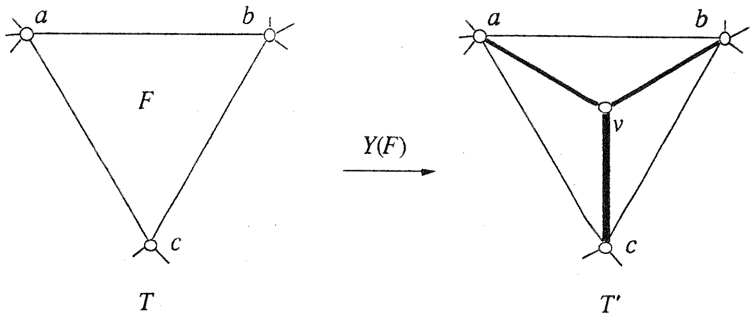

FIGURE 3. *Triangulating a face results in three braced edges incident with a cycle of three unbraced edges*

All faces of $T(4)$ are triply braced. Triangulating any one of its faces yields $T(5)$, which therefore has 6 braced edges and 3 unbraced edges (none of which is neutral). For higher orders we have

LEMMA 1. *Suppose there is a plane triangulation of order $n \geq 5$ with r braced edges and at least one doubly braced face. Then for any $k \geq 0$ there is a plane triangulation of order $n+k$ with $r+k$ braced edges.*

Proof. Triangulating a doubly braced faced results in a nett gain of one braced edge. Each new face is doubly braced, so iteration is possible. ¤

To discuss some applications of Lemma 1, it is convenient now to introduce some special notation. For any plane triangulation T, let $[T]$ denote the ordered pair (n,r), where n is the order of T and r is the number of braced edges in T. Let $Y_m T$ denote the operation of triangulating some face of T which is incident with exactly m braced edges ($m=0,1,2$ or 3). The latter notation is not completely unambiguous, because we do not explicitly specify which face is being triangulated, but this reflects the general possibility of more than one choice for the face, and so of more than one resultant plane triangulation.

A suitable starting configuration for Lemma 1 is $T(5)$. We have $\left[Y_2^k T(5)\right] = (k+5,\ k+6)$ for $k \geq 0$. By first triangulating one or more of the 8 unbraced faces of $T(6,1)$ we obtain $\left[Y_2^k Y_0^s\, T(6,1)\right] = (k+s+6,\ k+3s)$ for $k \geq 0$ and $1 \leq s \leq 8$. By first triangulating a singly braced face of $T(6,2)$ the method of Lemma 1 gives $[Y_2^k Y_1\ T(6,2)] = (k+7,\ k+9)$ for any $k \geq 0$. Alternatively $T(6,3)$ could be used instead of $T(6,2)$. Also $Y_1 T(6,2)$ has an unbraced face, so we can triangulate it to obtain $[Y_2^k\ Y_0\ Y_1 T\ (6,2)] = (k+8,\ k+12)$ for $k \geq 0$.

Some rather elusive pairs of parameters not achieved by the families of triangulations already discussed are attained by refinements of $Y_2 T(6,2)$. This triangulation has two pairs of singly braced faces, each pair sharing a braced edge. Hence we have $[Y_1^2 Y_2^k T(6,2)] = (k+8,\ k+1))$ for $k \geq 1$. Each triangulation of a singly braced face converts the adjacent singly braced face to an unbraced face, so we can obtain $[Y_0 Y_1^2 Y_2^k T(6,2)] = (k+9,\ k+14)$ and $[Y_0^2 Y_1^2 Y_2^k T(6,2)] = (k+10,\ k+17)$ for $k \geq 1$.

4. Subdividing an Edge

Another way to refine a plane triangulation T of order $n \geq 4$ into a plane triangulation T' of order $n+1$ is by *subdividing an edge*. If ab is any edge of T, the operation of subdividing ab will be denoted by $X(ab)$. Let F and F' be the two faces of T incident with ab, let $R := F \cup F'$, and let the boundary cycle of the region R be $C := \partial R :=$ *acbda*. Applying $X(ab)$ deletes ab from T, and inserts a new vertex v and four new edges av, bv, cv and dv in R (Figure 4). Straightforward analysis (we omit the details) shows that if ab is unbraced in T, then $X(ab)$ replaces it with four edges which are

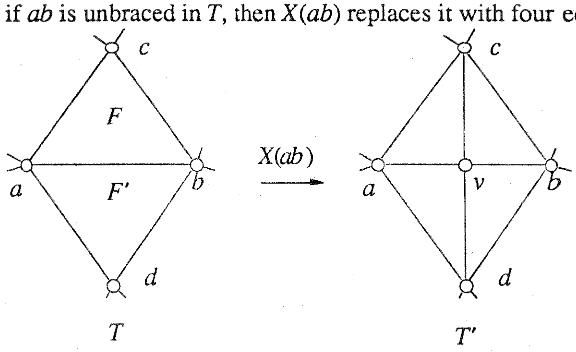

FIGURE 4. *Subdividing an edge: if it was unbraced, all four new faces are unbraced.*

unbraced and all edged of C are unbraced in T'. In other words, all four new faces are unbraced in T'. Moreover any edge which is braced by ab in T is unbraced in T'. On the other hand, if ab is braced in T, then $X(ab)$ replaces it by two unbraced and two braced edges in T', and all edges of C are unbraced in T' unless there is a face in T with boundary *acda* or *bcdb*.

LEMMA 2. *Suppose there is a plane triangulation of order n with r braced edges and at least one neutral edge incident with two unbraced faces. Then for any $k \geq 0$ there is a plane triangulation of order $n+k$ with r braced edges.*

Proof. Subdividing a neutral edge does not create any braced edges. There is no edge being braced by a neutral edge, so its removal does not change the status of any braced edge. The edges of the faces incident with the neutral edge are unbraced, so subdividing the neutral edge does not change their status. And the four new edges created by the subdivision are all neutral, so iteration is possible. ¤

We shall use X_0T to denote the operation of subdividing a neutral edge of T which is incident with two unbraced faces. Starting with $T(6,1)$, we have $[X_0^kT(6,1)] = (k+6,0)$ for $k \geq 0$. More generally, we can combine this construction with subdivision of unbraced faces: since $X_0^kT(6,1)$ has $2k+8$ faces, all unbraced (indeed, all edges are neutral), we have $[Y_0^sX_0^kT(6,1)] = (k+s+6,\ 3s)$ for $k \geq 0$ and $0 \leq s \leq 2k+8$. If we ensure that there is at least one doubly braced face present, we can also use the method of Lemma 1 to achieve $[Y_2^tY_0^sX_0^kT(6,1)] = (k+s+t+6,\ 3s+t)$ for $k \geq 0$, $1 \leq s \leq 2k+8$ and $t \geq 0$.

5. Orders for Given Numbers of Braced Edges

By triangulating faces and subdividing edges, as described in Sections 3 and 4, we can pass from low order starting configurations to all higher orders by constructions which show that for $r = 0$ and for each $r \geq 3$ there is an integer n_r such that for every $n \geq n_r$ there is a plane triangulation T with $[T] = (n,r)$.

To fill the gap at $r=1$ and $r=2$ we need starter plane triangulations with just one or two braced edges. In Figure 5 we show a plane triangulation $T(10)$ with just one braced edge, ab. Subdivide the braced edge ab; the resulting plane triangulation $T(11) := X(ab)\,T(10)$ has just two braced edges.

Now the method of Lemma 2 achieves $[X_0^kT(10)] = (k+10,1)$ and $[X_0^kT(11)] = (k+11,2)$ for $k \geq 0$. Of course $Y_2^tY_0^sX_0^kT(\mathrm{m})$, for m=10 or 11 and suitable values of k, s and t, gives a more extensive family of plane triangulations but the additional values of the parameters (n,r) only duplicate those already achieved by earlier constructions. Gathering together results now achieved, we have proved

THEOREM 1. *For every integer $r \geq 0$ there is an integer n_r such that for every $n \geq n_r$ there is a plane triangulation of order n with exactly r braced edges.*

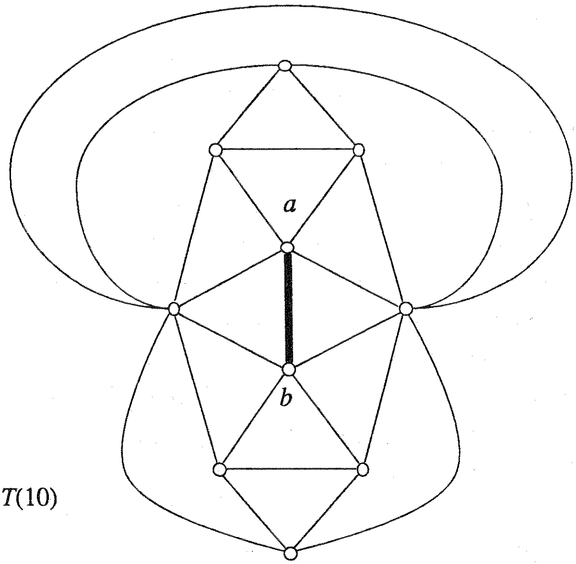

FIGURE 5. *A plane triangulation of order 10 with one braced edge.*

COROLLARY. *Suitable values of n_r are n_0=6, n_1=10, n_2=11;*

$$n_r = \left\lfloor \frac{r+1}{3} \right\rfloor + 7 \text{ for } 3 \leq r \leq 25; \quad n_r = \left\lfloor \frac{r+1}{2} \right\rfloor + 3 \text{ for } r \geq 26.$$

Proof. These values are provided by the constructions described; those for $r \geq 3$ are derived from $\left[Y_2^t\, Y_0^s\, X_0^k\, T(6,1)\right] = (k+s+t+6,\ 3s+t)$ with $k \geq 0$, $1 \leq s \leq 2k+8$ and $t \geq 0$. ¤

The best values we know for n_r, provided by the constructions we have described, are given by the following brief table ($r \leq 26$):

r	n_r	r	n_r	r	n_r
0	6	9	7	18	11
1	10	10	8	19	12
2	11	11	9	20	13
3	7	12	8	21	13
4	8	13	9	22	14
5	9	14	10	23	15
6	8	15	10	24	14
7	9	16	11	25	15
8	10	17	12	26	16

6. Local Constraints on Braced Edges

The ways in which braced edges can occur in a plane triangulation turn out to be restricted by certain aspects of local structure. We shall now consider some of these restrictions.

LEMMA 3. *No face of a plane triangulation of order $n \geq 5$ is triply braced.*

Proof. Suppose F is a triply braced face in a plane triangulation T of order $n \geq 4$, and let the boundary of F be the cycle $\partial F := abca$. Let the second face incident with the edge ab be F', with boundary cycle $\partial F' := abda$. Then $\partial F' \neq \partial F$ since $n > 3$, so $d \notin \{a,b,c\}$. Since ab is braced, there is an edge cd in T. The cycle $C := acbda$ is the boundary of the region $R := F \cup F'$, so separates the edges ab and cd, by the Jordan Curve Theorem. Let the second face incident with ac be F'', with boundary cycle $\partial F'' := acxa$. Since ac is braced, then is an edge bx in T. But the cycle $C' := acda$ separates the faces F and F'', yet it must not separate the endpoints of the edge bx. The only way C' can fail to separate b and x is that x is a vertex of C', so $x=d$. Similarly, if F''' is the second face incident with bc, and y is its third vertex, the fact that bc is braced forces $y=d$. But now $F \cup F' \cup F'' \cup F'''$ is the whole plane, so $n=4$. Hence F cannot exist if $n \geq 5$. ¤

LEMMA 4. *In a plane triangulation, all edges incident with a vertex of degree 3 are braced.*

Proof. If a is a vertex of degree 3 in a plane triangulation T, it must be incident with three faces F, F', F'' with boundaries $\partial F := abca$, $\partial F' := acda$ and $\partial F'' := abda$ (Figure 6). The edges ab, ac and ad are braced by cd, bd and bc respectively. ¤

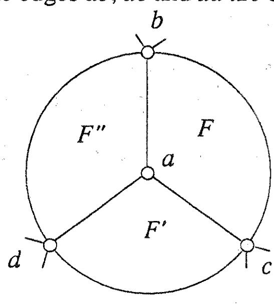

FIGURE 6: *Neighbourhood of a vertex of degree 3*

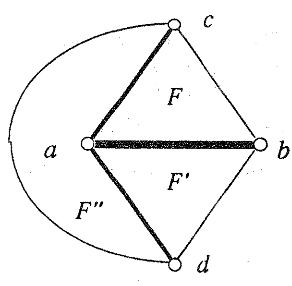

FIGURE 7: *A doubly braced face F*

LEMMA 5. *Any doubly braced face of a plane triangulation of order $n \geq 5$ is incident with a vertex of degree 3.*

Proof: Let F be a doubly braced face in a plane triangulation T of order $n \geq 5$, with boundary $\partial F := abca$, and let ab and ac be the braced edges of F. Arguing as in the proof of Lemma 3, T must contain the faces F' and F'' with boundaries $\partial F' := abda$ and $\partial F'' := acda$ (Figure 7).Then we have located all faces incident with a, so deg(a) = 3. Hence F is incident with a vertex of degree 3. ¤

Lemmas 4 and 5 show that doubly braced faces occur in triples in a plane triangulation, and correspond to faces created by triangulating a face of a lower order plane triangulation. Also a singly braced face must have its braced edge incident with another singly braced face, by Lemmas 4 and 5, so singly braced faces occur in pairs, with a common braced edge.

It is convenient to define an *isolated braced edge* in a plane triangulation to be a braced edge which is not adjacent to any other braced edge.

LEMMA 6. *Any plane triangulation with an isolated braced edge has order $n \geq 10$. Moreover, $T(10)$ is the unique triangulation of order $n = 10$ with only one braced edge.*

Proof. Suppose ab is an isolated braced edge in a triangulation T. Then ab is incident with 2 singly–braced faced: F_1 with boundary $abca$ and F_2 with boundary $abda$, where cd is an edge in T. Since ad is unbraced, Lemma 4 implies that deg(a) ≥ 4. Thus there must be another vertex interior to face F_1. But then to triangulate F_1 using vertices of degree at least 4 requires at least 3 vertices interior to F_1.

Since bd is unbraced, the same reasoning shows that face F_2 must be triangulated using at least 3 new vertices. So T must contain at least 10 vertices.

Furthermore, since the triangulations of faces F_1 and F_2 using vertices of degree 4 and unbraced edges are unique, the triangulation T is uniquely determined. It is $T(10)$ as shown in Figure 5. ¤

In view of our discussion of triangulations of small order, the values $n_0 = 6$ and $n_3 = 7$ included in the table in Section 5 clearly give the smallest possible orders for plane triangulations with $r = 0$ and $r = 3$ braced edges, respectively. We can now show that the surprisingly large tabulated values $n_1 = 10$ and $n_2 = 11$ are also smallest possible. That $n_1 = 10$ is smallest possible is immediate from Lemma 6.

An argument similar to the proof of Lemma 6 shows that if T has exactly two braced edges and these are adjacent, then T must have order $n \geq 11$. As previously noted, subdivision of the braced edge in $T(10)$ produces a triangulation of order $n = 11$ with $r = 2$, in which the braced edges are adjacent. Finally, the uniqueness of $T(10)$ as shown in Lemma 6 shows that any plane triangulation with $r = 2$ braced edges which are not adjacent must have order $n \geq 11$. (In fact, there is such a triangulation with order $n = 11$). Hence $n_2 = 11$ is smallest possible.

7. Maximal Numbers of Braced Edges

We have shown that any fixed number of braced edges can be achieved by plane triangulations of all sufficiently large orders. Let us now fix the order, and consider just how many braced edges can be achieved.

THEOREM 2: *Among all plane triangulations of order $n \geq 5$, the maximum number of braced edges is $2n-4$ if $n \equiv 2 \pmod 3$ and $2n-5$ otherwise.*

Proof. Suppose T has order $n \geq 5$ and maximum number r of braced edges. Each face is incident with at most 2 braced edges, by Lemma 3. Each edge is incident with two faces, and there are $2n-4$ faces, so $r \leq 2n-4$. Equality is achieved just if every face is doubly braced: Lemmas 4 and 5 show this happens if and only if T results from triangulating every face of some lower order plane triangulation T'. If T' has order $m \geq 3$, it has $2m-4$ faces, so triangulating them all to form T requires $n = 3m - 4 \equiv 2 \pmod 3$, $n \geq 5$, and $r = 3(2m-4) = 2n-4$.

In all other cases we must have $r < 2n - 4$, and some face of T is not doubly braced. If T has a singly braced face, its braced edge must be incident with two singly braced faces, by Lemmas 4 and 5. A suitable T with one pair of singly braced faces and all other faces are doubly braced results from triangulating all $2m-4$ faces of some plane triangulation T' of order $m \geq 3$, then triangulating one of the doubly braced faces to add one vertex and one braced edge to the total. Then $n = 3m - 3 \equiv 0 \pmod 3$, $n \geq 6$, and $r = 3(2m-4) + 1 = 2n - 5$.

If T has an unbraced face, and all other faces are doubly braced, it results from triangulating $2m-5$ faces of some plane triangulation T' of order $m \geq 4$. Then

$$n = 3m - 5 \equiv 1 \pmod 3),\ n \geq 7, \text{ and } r = 3(2m-5) = 2n - 5. \qquad \square$$

COROLLARY. *If G is a maximal planar graph of order $n \geq 5$, the number of vertices of degree 3 in G is at most* $\left\lfloor \frac{2n-4}{3} \right\rfloor$*, and this bound is achieved for each n.*

Proof. Any plane embedding of G is a plane triangulation T of order n. The edges incident with any vertex of degree 3 in T are all braced, by Lemma 4, so the number of vertices of degree 3 can be at most one third of the total number of braced edges. Attention to the residue classes of n modulo 3 justifies the upper bound $\left\lfloor \frac{2n-4}{3} \right\rfloor$ as a consequence of Theorem 2. The classes of plane triangulations constructed from lower order plane triangulations T' in the proof of Theorem 2 achieve this number of triangulated faces in all instances, and each triangulated face contains one vertex of degree 3. ¤

8. Maximal Configurations of Braced Edges

The plane triangulations invoked to prove Theorem 2 will now be shown to be the only classes to achieve maximum number of braced edges for fixed order. However, we can formulate the result more strongly with the help of another lemma.

LEMMA 7. *A plane triangulation of order* $n \equiv 2 \pmod 3$ *cannot have exactly* $2n - 5$ *braced edges.*

Proof. Let T be a plane triangulation of order $n \equiv 2 \pmod 3$, and suppose T has exactly $2n - 5$ braced edges. Put $k := (n-2)/3$, so the number of braced edges is $6k-1$.

It follows from Lemma 3 that no face is triply braced, and from Lemmas 4 and 5 that doubly braced faces occur in triples, and singly braced faces occur in pairs. Let m be the number of triples of doubly braced faces, p be the number of pairs of singly braced faces and q be the number of unbraced faces. The number of faces is $2n - 4 = 6k = 3m + 2p + q$. The number of braced edges is $2n - 5$ - $6k - 1 = 3m + p$, so $p + q = 1$.

Also $6k - 1 = 3m + p$ implies $p \equiv 2 \pmod 3$. Then $p + q = 1$ requires that one of p and q is negative, which is impossible. Thus T does not exist. ¤

It is convenient to introduce terminology for some special classes of plane triangulation. A *triangulated triangulation* is a plane triangulation T derivable from a lower order plane triangulation T' by triangulating every face. For example, $T(5)$ is a triangulated triangulation derived from $T(3)$. An *under–triangulated triangulation* results from triangulating all but one face of a lower order plane triangulation T', and an *over–triangulated triangulation* results from triangulating all faces of T' and then triangulating exactly one new face. For example, $T(4)$ is an under–triangulated triangulation of $T(3)$, while $T(6,2)$ and $T(6,3)$ are over–triangulated triangulations of $T(3)$. With this terminology we can characterize the triangulations which achieve the maximum number of braced edges.

THEOREM 3. *Any plane triangulation of order n with $2n-4$ braced edges is a triangulated triangulation with $n \equiv 2 \pmod 3$. Any plane triangulation of order n with $2n-5$ braced edges is either an under–triangulated triangulation with $n \equiv 1 \pmod 3$ or an over–triangulated triangulation with $n \equiv 0 \pmod 3$.*

Proof. Let T be a plane triangulation of order n with m triples of doubly braced faces, p pairs of singly braced faces, and q unbraced faces. If $[T] = (n, 2n-4)$ then $n \geq 5$, so Lemma 3 applies and the number of faces is $2n-4 = 3m + 2p + q$. The number of braced edges is $2n-4 = 3m + p$, so $p + q = 0$, whence $p = q = 0$ and $2n-4 = 3m$, whence $n \equiv 2 \pmod 3$. It follows that every face is doubly braced, and the faces occur in triples corresponding to triangulated faces of a lower order plane triangulation. Hence T is a triangulated triangulation.

If $[T] = (n, 2n-5)$, once again $n \geq 5$. Hence the number of faces is $2n-4 = 3m + 2p + q$ as before, and the number of braced edges is $2n-5 = 3m + p$, so $p + q = 1$. This time $n \equiv 2 \pmod 3$ is impossible, by Lemma 6. Suppose $n \equiv 1 \pmod 3$. Then $p \equiv 2n-5 \equiv 0 \pmod 3$, whence $p + q = 1$ implies $p = 0, q = 1$. Therefore T has one unbraced face, and all other faces are doubly braced and occur in triples corresponding to triangulated faces of a lower order plane triangulation. Hence T is an under–triangulated triangulation. It remains to consider the case $n \equiv 0 \pmod 3$. Then $p \equiv 2n-5 \equiv 1 \pmod 3$, whence $p + q = 1$ implies $p = 1, q = 0$. Therefore T has an adjacent pair of singly braced faces, and all other faces are doubly braced and occur in triples. It follows that T is an over–triangulated triangulation. ¤

9. Numbers of Braced Edges for Fixed Order

In Section 7 we established the maximum number of braced edges achievable by plane triangulations of fixed order. Now we complete the picture by showing which smaller numbers of braced edges are achieved among plane triangulations of fixed order.

THEOREM 4. *For each* $n \geq 11$ *there is a plane triangulation of order n with exactly r braced edges if and only if* $r \in \{0,1,2, \ldots, 2n-6\} \cup \{2n-q\}$, *where* $q := 5$ *if* $n \equiv 0$ *or* 1 (mod 3) *and* $q := 4$ *if* $n \equiv 2$ (mod 3).

Proof. The plane triangulations constructed in Sections 3 and 4 include at least one T with $[T] = (n+k, r)$ for

$$n = 11, k \geq 0 \text{ and } 0 \leq r \leq 16 \text{ or } r = 18$$
$$n = 12, k \geq 0 \text{ and } 0 \leq r \leq 19$$
$$n = 13, k \geq 0 \text{ and } 0 \leq r \leq 21.$$

Together, Theorem 2 and Lemma 6 show that the claimed range of values of r is best possible. We also know that any plane triangulation with exactly $r = 2$ braced edges has order $n \geq 11$, so the claimed result is best possible with regard to the claimed range of values of n. ¤

References

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