On Defining Sets of Directed Designs*

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Abstract

The concept of defining set has been studied in block designs and, under the name critical sets, in Latin squares and Room squares. Here we study defining sets for directed designs. A t-(v, k, λ) directed design (DD) is a pair $(V, \mathcal{B})$, where V is a v-set and $\mathcal{B}$ is a collection of ordered blocks (or k-tuples of V), for which each t-tuple of V appears in precisely λ blocks. A set of blocks which is a subset of a unique t-(v, k, λ)DD is said to be a *defining set* of the directed design.

As in the case of block designs, finding defining sets seems to be a difficult problem. In this note we introduce some lower bounds for the number of blocks in smallest defining sets in directed designs, determine

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the precise number of blocks in smallest defining sets for some directed designs with small parameters and point out an open problem relating to the number of blocks needed to define a directed design as compared with the number needed to define its underlying undirected design.

1 Introduction and some basic results

Let v, k, t, λ be integers such that $0 < t < k < v$ and $\lambda > 0$, and let V be a set of v elements. In this note, by a *k-tuple* of V, we mean a k-subset of V, ordered in the sense that the k-tuple $(a_1, a_2, \cdots, a_k)$ contains all and only the t-tuples of V $(a_{i_1}, a_{i_2}, \cdots, a_{i_t})$ with $i_1 < i_2 < \cdots < i_t$. Each k-tuple of V is called a *block*. In other words, a t-tuple is said to *appear in* a k-tuple if its components are contained in that block as a set, and if they are written in the same order. For example the ordered triples abc, abd, acd and bcd, but not the ordered triple acb, appear in the 4-tuple $abcd$.

A t-(v, k, λ) *directed design* (or simply a t-(v, k, λ)DD) is a pair $(V, \mathcal{B})$, where V is a v-set, and $\mathcal{B}$ a collection of blocks, such that each t-tuple of V appears in precisely λ blocks of $\mathcal{B}$. Directed designs were introduced in 1973 by Hung and N. S. Mendelsohn [10] who dealt with the case where $k = 3$. For further information see the survey papers by Colbourn and Rosa [5], Bennett and Mahmoodi [1] and E. Mendelsohn [17].

If the ordering of the blocks of a t-(v, k, λ)DD is ignored, then the unordered blocks form an underlying t-design. A straightforward counting argument shows that this t-design has parameters t-$(v, k, \lambda t!)$. Thus the 2-$(6, 3, 1)$, 2-$(7, 3, 1)$ and 2-$(7, 4, 1)$DDs (dealt with in Section 3) have underlying t-designs with parameters 2-$(6, 3, 2)$, 2-$(7, 3, 2)$ and 2-$(7, 4, 2)$ respectively.

Every t-(v, k, λ)DD is a $(t-1)$-(v, k, λ')DD, where $\lambda' = \lambda t(v - t + 1)/(k - t + 1)$ [1]. Consequently every t-(v, k, λ)DD is an s-(v, k, λ_s)DD, for $0 \le s \le t - 1$ where

$$\lambda_s = \lambda t! \binom{v-s}{t-s} \Big/ s! \binom{k-s}{t-s}. \tag{1}$$

Equation 1 shows that necessary conditions for the existence of a t-(v, k, λ)DD are that each λ_s is an integer. It has been shown (Bennett, Mahmoodi, Wei and Yin [2], Seberry and Skillicorn [18], D. J. Street and Seberry [22], D. J. Street and Wilson [23]) that when $t = 2$ and $k = 3, 4, 5$ or 6, these necessary conditions are also sufficient, except in two cases: neither a 2-$(15, 5, 1)$DD nor a 2-$(21, 6, 1)$DD exists. It has also been shown (Soltankhah [19], Grannell, Griggs and Quinn [8]) that when $t = 3$ and $k = 4$ the necessary conditions are again sufficient.

A set of blocks which is a subset of a unique t-(v, k, λ)DD is said to be a *defining set* of the directed design, denoted by $d^*(t\text{-}(v, k, \lambda))$. For example, the set of blocks $R = \{123, 214\}$ can be completed to a 2-$(4, 3, 1)$DD in two ways: by adjoining either $\{431, 342\}$ or $\{341, 432\}$. Hence R is not a defining set of either design. But the set of blocks $S = \{123, 431\}$ is a defining set of a directed design with the blocks $\{123, 431, 214, 342\}$.

A *minimal* defining set is a defining set, no proper subset of which is a defining set. A *smallest* defining set, denoted by $d_s^*(t\text{-}(v,k,\lambda))$, is a defining set which has smallest cardinality. Every t-(v,k,λ)DD has a defining set (the whole design) and hence a smallest defining set. A $d^*(t\text{-}(v,k,\lambda))$ consisting of blocks of a particular t-(v,k,λ) directed design D is denoted by d^*D and a smallest defining set by d_s^*D.

A (v,k,t) *directed trade* (or simply a (v,k,t)DT) of volume s consists of two disjoint collections T_1 and T_2, each of s blocks, such that every t-tuple of V is covered by precisely the same number of blocks of T_1 as of T_2. Such a DT is usually denoted by $T = T_1 - T_2$. Blocks in T_1 (T_2) are called the positive (respectively, negative) blocks of T. If $D = (V, \mathcal{B})$ is a directed design, and if $T_1 \subseteq \mathcal{B}$, we say that D *contains* the directed trade T. We often find it convenient to call a block of D which contains the element x an x–block.

In 1984, Curtis [6] found a smallest defining set for the unique Witt design with parameters 5-$(24, 8, 1)$ but the concept of defining sets for block designs in general was introduced by K. Gray [9]; recent surveys include [20], [21]. A similar idea has been studied in Latin squares and Room squares under the name *critical sets*; see for example Donovan and Hoffman [7], van Rees and Bate [24], the survey by Keedwell [11], Chaudhry and Seberry [3]. Defining sets for vertex colourings and edge colourings of graphs have also been studied; see for instance [11], Mahmoodian [13] and Mahmoodian, Naserasr and Zaker [14].

Here we consider defining sets for directed designs (as suggested by A. P. Street in [15]) beginning with propositions similar to those of K. Gray [9] for block designs.

Proposition 1. *Let $D = (V, \mathcal{B})$ be a t-(v,k,λ) directed design and let $S \subseteq \mathcal{B}$. Then S is a defining set of D if and only if S contains a block of every (v,k,t) directed trade $T = T_1 - T_2$ such that T is contained in D.*

Proof. First, suppose that S is a defining set of D and that $T = T_1 - T_2$ is a directed trade such that $T_1 \subseteq \mathcal{B}$ and $S \cap T_1 = \emptyset$. Then $S \subseteq \mathcal{B} \setminus T_1$ and the directed designs $(\mathcal{B} \setminus T_1) \cup T_i$, for $i = 1, 2$, are a pair of distinct t-(v,k,λ)DDs containing S. This contradicts the assumption that S is a defining set.

Next, suppose that S intersects every directed trade T contained in D, and that S is not a defining set of D. Then $S \subseteq D'$ for some t-(v,k,λ) directed design $D' = (V, \mathcal{B}')$ distinct from D. Hence $R = \mathcal{B} \setminus S$ must contain the same t-tuples as $R' = \mathcal{B}' \setminus S$ and there must exist a directed trade $T = T_1 - T_2$, where $T_1 \subseteq R$, $T_2 \subseteq R'$. But now $S \cap T_1 = \emptyset$, contradicting the assumption that S intersects every directed trade contained in D. □

We also have the following related result.

Proposition 2. *Let $D = (V, \mathcal{B})$ be a t-(v,k,λ) directed design and let $T_1 \subseteq \mathcal{B}$. If T_1 contains a block of every defining set of D then, for some T_2, $T = T_1 - T_2$ contains a directed trade.*

Proof. Since $\mathcal{B} \setminus T_1$ contains no defining set of D, it can be completed in at least two ways to directed designs with the same parameters as D: first, to D itself by taking $(\mathcal{B} \setminus T_1) \cup T_1$; secondly, to D' by taking $(\mathcal{B} \setminus T_1) \cup T_2 = \mathcal{B}'$. But now T_1 and T_2 contain the same t-tuples, and $T = T_1 - T_2$ contains a directed trade. □

Now every permutation on the elements of V induces a mapping from one k-tuple to another. An automorphism of a set of blocks X is a permutation of the elements of V which induces a permutation of the blocks of X. Let $Aut(X)$ denote the group of all automorphisms of X.

Proposition 3. *Suppose S is a particular defining set of a t-(v,k,λ) directed design D, and $\alpha \in Aut(D)$. Then $\alpha(S)$ is a defining set of D and $Aut(S)$ is a subgroup of $Aut(D)$.*

Proof. Let α be an automorphism of D, so that $\alpha(D) = D$. Clearly, if S is a defining set of D then $\alpha(S)$ is also a defining set of D.

Suppose α^* is any automorphism of S. Since $S \subseteq \mathcal{B}$, we have $\alpha^*(S) \subseteq \alpha^*(\mathcal{B})$. So $\alpha^*(S) = S$ is a subset of $\mathcal{B}$ and of $\alpha^*(\mathcal{B})$. But, since D is a t-(v,k,λ)DD, so is $\alpha^*(D)$, and since S is a defining set, $\alpha^*(D) = D$. Hence α^* is an automorphism of D. □

Proposition 4. *For $k > 2$, no automorphism of a 2-$(v,k,1)$DD consists of a single transposition.*

Proof. Without loss of generality, suppose we have a 2-$(v,k,1)$DD on $\{1,\cdots,v\}$, and that this design is fixed under the permutation (12). The element 1 belongs to some block, say B, where $2 \notin B$. For some elements x and y, both different from 1, B must contain the ordered pair xy. Since (12) is an automorphism, B and $(12)B$ are two distinct blocks, each containing the ordered pair xy. This contradicts the fact that $\lambda = 1$. □

Corollary. *For $k > 2$, any $d^*(2\text{-}(v,k,1))$, S, has at least $v-1$ elements occurring in its blocks.*

Proof. If two elements x and y do not appear in any block of S, then we have $(xy) \in Aut(S)$. By Proposition 3, $(xy) \in Aut(D)$, where D is the unique 2-$(v,k,1)$DD containing S. This contradicts Proposition 4. □

Proposition 5. *Let $D_1 = (V, \mathcal{B}_1)$ and $D_2 = (V, \mathcal{B}_2)$ be directed designs with parameters t-(v,k,λ_1) and t-(v,k,λ_2) respectively, and let $D = (V, \mathcal{B}_1 \cup \mathcal{B}_2) = D_1 \cup D_2$ which is a directed design with parameters t-$(v,k,\lambda_1+\lambda_2)$. If S is a defining set of D, then*

$$|S| \geq |d_s^* D_1| + |d_s^* D_2|.$$

Proof. Suppose that $|S| < |d_s^* D_1| + |d_s^* D_2|$; we show there is a contradiction. The blocks of D_1 and D_2 partition the blocks of D and consequently of S, with n_j blocks of S in D_j, for $j = 1, 2$. Then $n_j < |d_s^* D_j|$ for at least one value of $j = 1$ or 2, and for this value of j (by the definition of smallest defining set) these n_j blocks belong to two distinct designs, say D_j and D'_j, with the same parameters as the design D_j. So S is a subset of two distinct t-$(v, k, \lambda_1 + \lambda_2)$DDs, namely $D_\ell \cup D_j$ and $D_\ell \cup D'_j$, where $\ell \in \{1, 2\} \setminus \{j\}$. Hence S is not a defining set of D. □

Corollary. *If each D_i is a t-(v, k, λ_i)DD for $i = 1, \cdots, n$ and if $D = \cup_{i=1}^n D_i$, then*

$$|d_s^* D| \geq \sum_{i=1}^n |d_s^* D_i|.$$

2 The case $k = 3$

In this section we introduce a lower bound for $|d_s^* D|$, where D is a directed design with $k = 3$. First we prove some lemmas which will be useful in our discussion.

Lemma 1. *Let D be a 2-$(v, 3, 1)$ directed design based on the set V and let $x \in V$. Then x appears in every positive block of some directed trade contained in D.*

Proof. Let $x \in V$ and consider all the x–blocks of D. Two cases may occur.
(a) First, there may be an element appearing next to x in two of these blocks, say for example xab and bax, or xab and axc for some elements $a, b, c \in V$. (There are six possibilities altogether.) Then we have a directed trade of x–blocks, such as

$$T_1 = \{xab, bax\}, T_2 = \{xba, abx\}$$

or

$$T_1 = \{xab, axc\}, T_2 = \{axb, xac\}.$$

(b) Otherwise no other element appears next to x in more than one x–block of D. In this case some of the x–blocks have a structure such as

$$xa_1a_2,\ a_1a_3x,\ xa_4a_3, \cdots,\ xa_\ell a_{\ell-1},\ a_\ell a_2 x$$

which forms a directed trade with the blocks

$$a_1a_2x,\ xa_1a_3,\ a_4a_3x, \cdots,\ a_\ell a_{\ell-1}x,\ xa_\ell a_2.$$

In either case the lemma is true. □

Corollary. *If S is a defining set for a 2-$(v, 3, 1)$ directed design D, then all the elements of V appear in the blocks of S.*

Proof. This follows from Lemma 1 by Proposition 1. □

Lemma 2. *Let S be a defining set of a 2-$(v,3,1)$ directed design, D. If an element x appears only once in S, then x is the second element of at most one block in $D \setminus S$.*

Proof. Suppose that a and b are the elements which appear in the sole x–block in S, and that cxd is a block in $D \setminus S$ in which x appears in the second place.

As in the proof of Lemma 1, consider when two elements may appear *next* to each other in the x–blocks of $D \setminus S$. The block dxc would form, with cxd, a trade disjoint from S and thus cannot occur. Neither can there be any other pair of elements which occur next to each other in more than one x–block of $D \setminus S$. Hence all the other x–blocks in $D \setminus S$ have the form

$$xx_1c,\ x_1x_2x,\ xx_3x_2, \cdots,\ xx_\ell x_{\ell-1},\ x_\ell wx$$

and

$$dy_1x,\ xy_2y_1,\ y_2y_3x, \cdots,\ y_{k-1}y_kx,\ xzy_k.$$

Suppose that $\{a,b\} \cap \{c,d\} = \emptyset$. Since V is finite, the elements w and z must be chosen from the set $\{a,b\}$. So x appears in at most one block as a second element. The cases in which c or d, or both, belong to the set $\{a,b\}$ can be dealt with similarly.
□

Lemma 3. *Let S be a defining set of a 2-$(v,3,1)$ directed design D. If two elements x and y appear only once each in S, then they must appear in different blocks of S.*

Proof. Suppose that x and y occur in the same block of S, together with the element z. Then the elements x and y must appear together in a block B of $D \setminus S$. We show that there exists a directed trade $T = T_1 - T_2$, with $T_1 \subseteq D \setminus S$, contradicting the fact that S is a defining set. There are two cases to be considered.
(a) The other element in B is z. Then without loss of generality some of the blocks in $D \setminus S$ are of the form

$$xa_1a_2,\ a_1a_3x,\ xa_4a_3, \cdots,\ xa_\ell a_{\ell-1},\ a_\ell a_2x$$

where $\ell \leq v-3$. As in the proof of Lemma 1, this implies that $D \setminus S$ contains all the positive blocks of a directed trade.
(b) The other element of B is $w \neq z$. Then without loss of generality the following four cases must be investigated:

$$\begin{array}{lll} (i) & xyz \in S, & yxw \in D \setminus S; \\ (ii) & xzy \in S, & ywx \in D \setminus S; \\ (iii) & xyz \in S, & ywx \in D \setminus S; \\ (iv) & xzy \in S, & yxw \in D \setminus S. \end{array}$$

We note that the cases with $zxy \in S$ are similar to cases (i) and (iii), and so on, since interchanging the first and last elements of each block gives another directed design with the same parameters. Again the case of $xyz \in S, wyx \in D \setminus S$ is similar to case (i), and the case of $xzy \in S, wyx \in D \setminus S$ is similar to case (iv). We show that each case leads to a contradiction.

(i) In this case, by Lemma 2, x does not appear as a second element in any other block of $D \setminus S$. Thus x appears once as the second element of a block of D, and $(r-1)/2$ times each as the first element of a block and as the third element, where r is the replication number of each element in D. But $r = v - 1$, so v is even. Now if y also appears as the second element in a block of $D \setminus S$, then a similar count shows that v must be odd, which is a contradiction. So neither x nor y appears as the second element in any other block of $D \setminus S$. This implies that all the blocks in $D \setminus S$ which contain either x or y (but not both) form the positive blocks of a directed trade (since applying the permutation (xy) to those blocks gives the negative blocks of the trade).

(ii) In this case x does not appear as a second element in any block of $D \setminus S$. For then, as in the proof of Lemma 2, $D \setminus S$ contains either the block axw or the block xwb, which is impossible. The same is true for the element y. Thus neither x nor y appears as the second element in any other block of $D \setminus S$. Now a directed trade can be constructed in $D \setminus S$ as follows: to any of the other $v-3$ x–blocks, say xab or abx, we take aby or yab respectively as negative blocks of the trade, and similarly for any of the other $v-3$ y–blocks.

(iii) As in case (ii), neither x nor y appears as the second element in any other block of $D \setminus S$. Thus x appears as the first element of a block $(v-1)/2$ times, implying that v is odd. But y appears $(v-2)/2$ times as the first element of a block, implying that v is even, a contradiction.

(iv) As in case (i), x does not appear as a second element in any other block of $D \setminus S$, so v must be even. Thus the other x-blocks are

$$wx_1x,\ xx_2x_1,\ x_2x_3x, \cdots,\ xx_\ell x_{\ell-1},\ x_\ell zx.$$

If $\ell < v-4$, then as in the second part of Lemma 1, a directed trade can be constructed in the remaining x-blocks of $D \setminus S$. So $\ell = v-4$, and all $v-3$ x-blocks not initially assumed have appeared in the set of blocks given above. Now if y does not appear as a second element in any other block of $D \setminus S$, then v must be odd which is impossible. Thus y must appear as a second element in one block of $D \setminus S$, the other y-blocks are the same as those in Lemma 2 and the elements y and z appear next to each other in a block of $D \setminus S$.

Now a directed trade can be constructed in $D \setminus S$ as follows: for each of the other $v-3$ x-blocks, take a corresponding block in which x is replaced by y; for each of the other $v-3$ y-blocks, take a corresponding block in which y is replaced by x; finally, for the block $x_{v-4}zx$ take the corresponding block $x_{v-4}yz$, and for the block which contains yz, say αyz, take the corresponding block αzx. □

The following theorem gives a lower bound for the number of blocks in a defining set of a 2-$(v,3,1)$DD.

Theorem. *Let D be a* 2-$(v,3,1)$*DD, then* $|d_s^*D| \geq v/2$.

Proof. Let S be a defining set of D and let $|S| = s$. Then by the Corollary to Lemma 1, there exist at most s elements each of which appears in precisely one block of S. By Lemma 1, each of the other $(v - s)$ elements appears in at least two blocks of S. Since there are only $3s$ entries in the blocks we have

$$s + 2(v - s) \leq 3s,$$

and the result follows. □

Note that the Lemmas of this section are not necessarily true if $k > 3$ as shown in the case of the 2–$(7, 4, 1)$DD of the next section.

3 Defining sets of some small directed designs

In this section we discuss the defining sets for small directed designs, starting with the smallest nontrivial case. For all the examples in this section, the method for finding the lower bound on the cardinality of a smallest defining set is similar to the techniques of integer programming used for example by Khodkar [12]. The upper bound on the cardinality of a smallest defining set is found by taking a subset of the set of blocks of the design and showing that it completes uniquely. In each table, the blocks of the defining sets are shown in boldface.

2-(4,3,1)DDs

It is easily shown that, up to isomorphism, there are three 2-(4,3,1)DDs, each having automorphism group of order 4 and smallest defining set of cardinality 2. Examples of each design are given in Table 1, together with generators of their automorphism groups.

123	**431**	214	342	(1324)
123	**341**	214	432	(12)(34), (13)(24)
123	**314**	241	432	(1243)

Table 1: The three 2-(4,3,1) DDs: smallest defining sets and group generators

2-(6,3,1)DDs

Given a 2-$(6, 3, 1)$DD on the set $V = \{1, \cdots, 6\}$, consider the positions of each element in the 10 blocks of this design. For the element x, let x_i be the number of appearances of x in position i, where $1 \leq i \leq 3$. Counting the number of ordered pairs containing x gives $2x_1 + x_2 = 5 = x_2 + 2x_3$, so that $x_1 = x_3 \in \{0, 1, 2\}$, $x_2 \in \{1, 3, 5\}$. This leads to three possible solutions:

$$(x_1, x_2, x_3) = (0, 5, 0) \text{ or } (1, 3, 1) \text{ or } (2, 1, 2). \tag{2}$$

Clearly for each fixed position i we have

$$\sum_{1 \leq x \leq 6} x_i = 10, \quad i = 1, 2, 3.$$

For $j \in \{1, 2, 3\}$, let a_j be the number of elements with frequencies as in the jth solution of Equation 2 above. Then

$$a_1 + a_2 + a_3 = 6$$

and we have for the first and third positions

$$a_2 + 2a_3 = 10$$

and for the second position

$$5a_1 + 3a_2 + a_3 = 10.$$

These three equations together give two solutions:

$$(a_1, a_2, a_3) = (0, 2, 4) \text{ or } (1, 0, 5). \tag{3}$$

Colbourn and Colbourn [4] showed that there are 32 2-(6,3,1)DDs. We find that 30 of these, all with trivial automorphism groups, correspond to the first solution of Equation 3, and that the remaining two, both with automorphism groups of order 5, correspond to the second. Of those corresponding to the first solution, designs $1, \cdots, 28$ in Table 2 have smallest defining sets of cardinality 5, whereas designs 29 and 30 have smallest defining sets of cardinality 4. Both of those corresponding to the second solution (designs 31 and 32) have smallest defining sets of cardinality 4. We note that the automorphism groups of designs 31 and 32 are generated by the permutations (13564) and (12463) respectively.

2-(7,3,1)DDs

Up to isomorphism, there are 2368 2-(7,3,1) directed designs [4] of which only 221 have non-trivial automorphism groups. As an example, we show one of them (the first listed with non–trivial automorphism group), with automorphism group generated by (16)(23)(47), and smallest defining set of cardinality 6.

123 **214** **315** **416** 517 265 427 632 734 **367** 453 752 **564** 761.

2-(7,4,1)DDs

It is shown in [16] that, up to isomorphism, there exist two 2-(7,4,1)DDs. Each has automorphism group of order 7 and smallest defining set of cardinality 2. Examples of each design are given in Table 3, together with generators of their automorphism groups.

Open question

The unique 2-(6,3,2) design has smallest defining set of cardinality three, but each of the 2-(6,3,1)DDs needs at least four blocks to form a defining set. The

#											$\lvert d_s^* \rvert$
1	123	**214**	315	416	425	**326**	534	**643**	**561**	**652**	5
2	123	**214**	315	416	425	**326**	534	**643**	**651**	**562**	5
3	123	**214**	315	416	425	**526**	**362**	534	**643**	**651**	5
4	123	**214**	315	416	425	**526**	**362**	**634**	543	**651**	5
5	123	**214**	315	416	425	**526**	**632**	**364**	543	**651**	5
6	123	**214**	315	416	**256**	**362**	534	452	**643**	**651**	5
7	123	**214**	315	416	**256**	**362**	634	452	**543**	**651**	5
8	123	**214**	315	416	**256**	**362**	634	**542**	453	**651**	5
9	123	**214**	315	416	**256**	**632**	364	**452**	543	**651**	5
10	123	**214**	315	416	**256**	**632**	364	**542**	453	**651**	5
11	123	**214**	315	416	**265**	**362**	**534**	452	643	**561**	5
12	123	**214**	315	416	**265**	**362**	634	452	**543**	**561**	5
13	123	**214**	315	416	**265**	**362**	634	**542**	453	**561**	5
14	123	**214**	315	416	**265**	**632**	364	452	**543**	**561**	5
15	123	**214**	315	416	**265**	**632**	364	**542**	453	**561**	5
16	123	**214**	315	416	**625**	**326**	**534**	452	643	**561**	5
17	123	**214**	315	416	**625**	**326**	634	452	**543**	**561**	5
18	123	**214**	315	416	**625**	**326**	634	542	**453**	**561**	5
19	123	**214**	315	**516**	265	**362**	**534**	**461**	452	643	5
20	123	**214**	315	**516**	265	**362**	**534**	**641**	452	463	5
21	123	**214**	315	**516**	265	**632**	**534**	436	**641**	452	5
22	123	**214**	315	**516**	265	**632**	346	**641**	452	**543**	5
23	123	**214**	315	**516**	**625**	**326**	534	461	452	**643**	5
24	123	**214**	315	**516**	**625**	**326**	534	641	452	**463**	5
25	123	**214**	315	**516**	**625**	**326**	634	461	452	**543**	5
26	123	**214**	315	**516**	**625**	**326**	634	461	542	**453**	5
27	123	**214**	**156**	**625**	**326**	351	634	461	542	**453**	5
28	123	**214**	**156**	**625**	**326**	531	634	**435**	461	542	5
29	123	**145**	316	241	**325**	**526**	**634**	462	543	651	4
30	123	**145**	316	241	**256**	**352**	**634**	462	543	651	4
31	123	**145**	316	**421**	624	**325**	**526**	534	463	651	4
32	123	**154**	316	**241**	256	**352**	634	462	453	**651**	4

Table 2: Smallest defining sets of the 32 2-(6,3,1)DDs

1234	**3156**	2617	7541	5372	6425	4763	(1357462)
1234	**4156**	5317	2761	6473	7542	3625	(1463572)

Table 3: The two 2-(7,4,1)DDs: smallest defining sets and group generators

unique 2-(7,4,2) design has smallest defining set of cardinality three, but each of the 2-(7,4,1)DDs needs only two blocks to form a defining set. On the other hand, each of the four 2-(7,3,2) designs has smallest defining set of cardinality six, and so does the 2-(7,3,1)DD considered above. In general, how does the cardinality of the smallest defining set of a directed design compare with that of its underlying undirected design?

References

[1] F. E. Bennett and A. Mahmoodi, *Directed designs*, in CRC Handbook of Combinatorial Designs, C. Colbourn and J. H. Dinitz, eds., Discrete Mathematics and its Applications, CRC Press, Boca Raton, New York, London, Tokyo, 1996, pp. 317–321.

[2] F. E. Bennett, A. Mahmoodi, R. Wei and J. Yin, *Existence of DBIBDs with block size six*, Utilitas Mathematica **43** (1993), 205–217.

[3] G. R. Chaudhry and J. Seberry, *Minimal critical set of a Room square of order* 7, Bulletin of the Institute of Combinatorics and its Applications **20** (1997), 90.

[4] C. J. Colbourn and M. J. Colbourn, *Some small directed triple systems*, Congressus Numerantium **30** (1981), 247–255.

[5] C. J. Colbourn and A. Rosa, *Directed and Mendelsohn triple systems*, in Contemporary Design Theory: a collection of surveys, J. H. Dinitz and D. R. Stinson, eds., Wiley Interscience Series in Discrete Mathematics and Optimization, 1992, pp. 97–136.

[6] R. T. Curtis, *Eight octads suffice*, Journal of Combinatorial Theory (Series A) **36** (1984), 116–123.

[7] D. Donovan and D. G. Hoffman, *Critical sets in a family of groups*, Australasian Journal of Combinatorics **16** (1997), 21–28.

[8] M. Grannell, T. S. Griggs and K. Quinn, *All possible* 3-$(v, 4, \lambda)$ *designs exist*, submitted for publication.

[9] K. Gray, *On the minimum number of blocks defining a design*, Bulletin of the Australian Mathematical Society **41** (1990), 97–112.

[10] S. H. Y. Hung and N. S. Mendelsohn, *Directed triple systems*, Journal of Combinatorial Theory (Series A) **14** (1973), 310–318.

[11] A. D. Keedwell, *Critical sets for latin squares, graphs and block designs: a survey*, Congressus Numerantium **113** (1996), 231–245.

[12] A. Khodkar, *Smallest defining sets for the* 36 *non-isomorphic twofold triple systems of order nine*, Journal of Combinatorial Mathematics and Combinatorial Computing **17** (1995), 209–215.

[13] E. S. Mahmoodian, *Some problems in graph colorings*, in Proceedings of the 26th Annual Iranian Mathematics Conference, (ed. S. Javadpour and M. Radjabalipour), Kerman, Iran, March 1995, Iranian Mathematical Society, University of Kerman, pp. 215–218.

[14] E. S. Mahmoodian, R. Naserasr and M. Zaker, *Defining sets of vertex colorings of graphs and latin rectangles*, Discrete Mathematics **167/168** (1997), 451–460.

[15] E. S. Mahmoodian and M. A. Shokrollahi, *Open problems at the combinatorics workshop of AIMC25 (Tehran,* 1994), in Combinatorics Advances, C. J. Colbourn and E. S. Mahmoodian, eds., Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995, pp. 321–324.

[16] E. S. Mahmoodian and N. Soltankhah, *Intersections of* 2-$(v, 4, 1)$ *directed designs*, Journal of Combinatorial Mathematics and Combinatorial Computing **20** (1996), 225–236.

[17] E. Mendelsohn, *Mendelsohn designs*, in CRC Handbook of Combinatorial Designs, C. Colbourn and J. H. Dinitz, eds., Discrete Mathematics and its Applications, CRC Press, Boca Raton, New York, London, Tokyo, 1996, pp. 388–393.

[18] J. R. Seberry and D. Skillicorn, *All directed BIBDs with* $k = 3$ *exist*, Journal of Combinatorial Theory (Series A) **29** (1980), 244–248.

[19] N. Soltankhah, *Directed quadruple designs*, in Combinatorics Advances, C. J. Colbourn and E. S. Mahmoodian, eds., Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995, pp. 277–291.

[20] A. P. Street, *Defining sets for block designs: An update*, in Combinatorics Advances, C. J. Colbourn and E. S. Mahmoodian, eds., Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995, pp. 307–320.

[21] A. P. Street, *Trades and defining sets*, in CRC Handbook of Combinatorial Designs, C. J. Colbourn and J. H. Dinitz, eds., Discrete Mathematics and Its Applications, CRC Press, Boca Raton, New York, London, Tokyo, 1996, pp. 474–478.

[22] D. J. Street and J. R. Seberry, *All DBIBDs with block size four exist*, Utilitas Mathematica **18** (1980), 27–34.

[23] D. J. Street and W. H. Wilson, *On directed balanced incomplete block designs with block size five*, Utilitas Mathematica **18** (1980), 161–174.

[24] G. H. J. van Rees and J. A. Bate, *The size of the smallest strong critical set in a latin square*, (submitted for publication).

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