Hall's condition yields less for multicolorings

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Abstract

It has been shown that a vertex list assignment satisfying Hall's condition is sufficient for the existence of a proper list coloring if and only if every block of the graph is a clique. It is asked in [5] whether this can be extended to multicolorings. It is shown here that in general this is not the case.

Introduction

The notion of list coloring has its origins independently with Vizing [7] and Erdös, Rubin & Taylor [1]. When the graph is a complete graph, list colorability with sets A_v available at each vertex v is equivalent to the set system $\{A_v : v \text{ is a vertex}\}$ having a *system of distinct representatives.* Thus when the graph is complete the conditions given by P. Hall [2] are sufficient for the existence of a list coloring. Hilton and Johnson [4] noticed this and introduced the idea of placing conditions on the lists which would imply list colorability. Because of the condition's similarity to those given by Hall, Hilton and Johnson dubbed it Hall's condition.

Suppose G is a finite simple graph, $L:V(G)\to\{\text{finite subsets of a set } S\}$ and $\kappa:V(G)\to\mathbb{N}$. G is said to be *(L,κ)-colorable* if there exist $A_v \subseteq L(v)$ with $|A_v| = \kappa(v)$ for every $v \in V(G)$ such that $A_u \cap A_v = \emptyset$ whenever $uv \in E(G)$. For each $\sigma \in S$ and induced subgraph H of G, let $\alpha(\sigma,L,H)$ denote the size of the largest independent set of vertices in H whose lists contain σ.

Definition. G, L and κ defined as above are said to satisfy *Hall's condition* if and only if for each induced subgraph H of G

$$\sum_{\sigma\in S}\alpha(\sigma,L,H) \geq \sum_{v\in V(H)}\kappa(v).$$

Note that Hall's condition is necessary for the existence of a proper (L,κ)-coloring.

Hilton and Johnson [4,6] have shown that when $\kappa = 1$ the following extension of Hall's Theorem holds.

Theorem 1. *G, L and κ satisfying Hall's condition is sufficient for the existence of a proper (L,κ)-coloring when $\kappa = 1$ if and only if every block of G is a clique.*

Halmos and Vaughn [3] generalized Hall's Marriage Theorem to the case where the set representatives are not just singletons, but are subsets of a given size. In other words, they showed that the existence of a proper list multicoloring of a complete graph is merely equivalent to a generalized Hall's Marriage Theorem.

We can derive Halmos and Vaughn's result from theorem 1 by taking G to be a complete graph, $L:V(G)\to\{$finite subsets of $S\}$ and $\kappa:V(G)\to\mathbb{N}$. We obtain an auxiliary graph G^* and list assignment $L^*:V(G^*)\to\{$finite subsets of $S\}$ by replacing each vertex $v\in V(G)$ with a $K_{\kappa(v)}$ and defining $L^*(u) = L(v)$ when u is a vertex of the $K_{\kappa(v)}$ inserted for the vertex v. We take $\kappa^* = 1$ and note that G^* is also a complete graph with G^*, L^* and κ^* satisfying Hall's condition if and only if G, L, and κ satisfy Hall's condition. For a complete account of this see [5].

The Counterexample

The idea used to derive Halmos and Vaughn's result when applied to graphs where every block is a clique when there is more than one block may cause cut-vertices to no longer be cut-vertices. Thus we may not be able to generalize Theorem 1 in this manner to arbitrary $\kappa:V(G)\to\mathbb{N}$. In fact, as the main intent of this exposition is to show, the general analogue to Theorem 1 does not hold. We next give an example to show that this is the case.

Example 2. Let G be the graph in the figure below. Also illustrated is a list function $L:V(G)\to\mathcal{P}(\{a,b,c,d,e,f\})$ and a function $\kappa:V(G)\to\mathbb{N}$. Although G, L and κ satisfy Hall's condition, there is no proper (L,κ)-coloring of G. So the analogue to Theorem 1 does not exist for multicolorings in general.

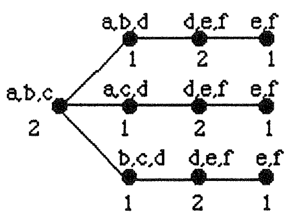

Figure 1.

In order to see that there is no proper (L,κ)-coloring of G we first declare the vertex of degree three to be denoted by v. Clearly any (L,κ)-coloring must use either {a,b},{a,c} or {b,c} at v. However, any of these choices forces one neighbor of v to be given {d} which forces its unique neighbor excluding v to be given {e,f}. This then leaves no choice for the pendant vertex on that branch. Thus we have that G can not be properly (L,κ)-colored.

We shall show that Hall's condition is satisfied by showing that

$$\sum_{\sigma}\alpha(\sigma,L,G) \geq \sum_{v\in V(G)}\kappa(v)$$

and that each vertex deleted subgraph can be properly (L,κ)-colored. It is quickly determined that

$$\sum_{\sigma}\alpha(\sigma,L,G) = 15 > \sum_{v\in V(G)}\kappa(v).$$

Thus, we need only show that each vertex deleted subgraph is (L,κ)-colorable.

It is obvious that $G-v$ can be properly (L,κ)-colored so we suppose that $w\in V(G)$, $w\neq v$ and consider the graph $G-w$. If x is the unique neighbor of v on the the branch connecting w to v, we may suppose by symmetry that $L(x) = \{a,b,d\}$. We now properly (L,κ)-color $G-w$ as follows. Assign {a,b} to vertex v, so the two branches not containing w in G can be properly (L,κ)-colored. It is easy to see that this coloring can be extended to the remaining vertices of $G-w$ resulting in $G-w$ being properly (L,κ)-colored.

Therefore G,L and κ satisfy Hall's condition and since every block of G is a clique we have shown that the analogue to Theorem 1 does not generally hold. ▯

Clearly the nature of the function κ is important in determining whether or not a (L,κ)-coloring exists. We can generalize Theorem 1 when κ satisfies a particular property.

Theorem 3. *Let G be a finite simple graph, L be a vertex list assignment and $\kappa: V(G)\rightarrow$ {finite subsets of S} such that $\kappa(v) = 1$ for all cut vertices $v\in V(G)$. Then G, L and κ satisfying Hall's condition is sufficient for the existence of a proper (L,κ)-coloring if and only if every block in G is a clique.*

Proof. Let G^* be obtained from G as was done previously. Since cut vertices remain cut-vertices and blocks of G^* are cliques if and only if the blocks of G are cliques, the result follows from theorem 1. ▯

This leads to the obvious problem of determining what other conditions, if any, placed on the function κ make Hall's condition sufficient for the existence of a proper (L,κ)-coloring of G.

References

[1] Erdös, Rubin & Taylor, *Choosability in Graphs.* (Proceedings of the West Coast Conference On Combinatorics, Graph Theory and Computing, 1979) Congressus Numerantium 26(1979),125-157.

[2] P. Hall, *On Representatives of Subsets.* Journal of London Math. Society 10(1935),26-30.

[3] Halmos & Vaughn, *The Marriage Problem.* Amer. Journal of Math., 72(1950), 214-215.

[4] A.J.W. Hilton & P.D. Johnson Jr., *Extending Halls Theorem.* "Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel", Physica-Verlag, Heidelberg, 1990, p. 359-371.

[5] A.J.W. Hilton, P.D. Johnson Jr. and D.A. Leonard, *Hall's Condition for Multicolorings.* Congressus Numerantium, to appear.

[6] A.J.W. Hilton, P.D. Johnson Jr. and E.B. Wantland, *The Hall Number of a Simple Graph.* Congressus Numerantium, to appear.

[7] V.G. Vizing, *Coloring the Vertices of a Graph in Prescribed Colors.* (Russian) Diskret. Analiz. No. 29 Metody Diskret. Anal. V. Teorii Kodov i Shem (1976) 3-10, 101(MR5816371).

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