Bounds on the Weak Domination Number

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Abstract

Let $G = (V, E)$ a graph. A set $D \subseteq V$ is a weak dominating set of G if for every vertex $y \in V - D$ there is a vertex $x \in D$ with $xy \in E$ and $d(x, G) \leq d(y, G)$. The weak domination number $\gamma_w(G)$ is defined as the minimum cardinality of a weak dominating set and was introduced by Sampathkumar and Pushpa Latha in [6].

In this paper we present sharp upper bounds on $\gamma_w(G)$ for general graphs involving the maximum and minimum degree and characterize all extremal graphs. Furthermore, we give a probabilistic upper bound and a lower bound on $\gamma_w(G)$.

1. Introduction

In this paper we only consider finite, undirected graphs G without multiple edges or loops. Let G be such a graph. $V(G)$ and $E(G)$ denote the vertex set and edge set of G. The neighbourhood of any vertex $x \in V(G)$ in a graph G is denoted by $N(x, G)$. For any subset $X \subseteq V(G)$ we set $N(X, G) = \cup_{x \in X} N(x, G)$ and $N[X, G] = X \cup N(X, G)$. The degree $d(x, G)$ of a vertex $x \in V(G)$ in the graph G is the cardinality of its neighbourhood $|N(x, G)|$. The minimum (maximum) degree $\delta(G)$ ($\Delta(G)$) of a graph G is defined as $\delta(G) = \min\{d(x, G) | x \in V(G)\}$ ($\Delta(G) = \max\{d(x, G) | x \in V(G)\}$).

Whenever it is clear to which graph G we refer, we only write $N(x)$ ($N(X)$, $N[X]$, $d(x)$ respectively) instead of $N(x, G)$ ($N(X, G)$, $N[X, G]$, $d(x, G)$ respectively).

For every vertex $v \in V(G)$ we define the **weak degree** $d_w(v, G)$ of v in G as

$$d_w(v, G) = |\{u \mid u \in N(v, G), d(u, G) \leq d(v, G)\}|,$$

i.e. the weak degree of v is the number of neighbours of v which have at most the same degree as v. Let $W(G)$ denote the set of all vertices v of G with $d_w(v, G) = 0$. Clearly, $W(G)$ is an independent set.

A **weak dominating set** of a graph G is a set D with the property that for all vertices $y \in V(G) - D$ there is a vertex $x \in N(y, G) \cap D$ with $d(y, G) \geq d(x, G)$, i.e. every vertex not in D is dominated by a vertex in D having at most the same degree. In this situation we say that x **is a weak neighbour of** y. By this definition, it is clear that $W(G) \subseteq D$ for any weak dominating set D of G. The **weak domination number** $\gamma_w(G)$ of a graph G is defined as the minimum cardinality of a weak dominating set of G.

This parameter was introduced by E. Sampathkumar and L. Pushpa Latha in [6] and represents one of the numerous versions of the classical domination number $\gamma(G)$ which have been introduced and studied in the recent years.

Apart from the immediate bounds $\gamma(G) \leq \gamma_w(G) \leq n(G) - \delta(G)$ for graphs G of order $n(G)$ no upper bound on $\gamma_w(G)$ has been published so far. Considering the star $K_{1,n-1}$ with $\gamma_w(K_{1,n-1}) = n-1$ it is easy to understand the difficulties of finding a reasonable upper bound on $\gamma_w(G)$. Even for such a simple graph a minimum weak dominating set may contain almost all vertices.

But if the ratio $\frac{\gamma_w(K_{1,n-1})}{n}$ tends to 1 the maximum degree $n-1$ must tend to infinity and the minimum degree remains 1. It was this observation which lead to the results presented in Section 3. We begin with some preparatory results in Section 2. A probabilistic argument leads to the upper bound in Section 4 and in Section 5 we give a lower bound on $\gamma_w(G)$. Further results on $\gamma_w(G)$ can be found in [3] and [4].

2. Preliminary Results

In [6] the authors also introduced a **strong domination number** $\gamma_{st}(G)$ which differs from $\gamma_w(G)$ only by the opposite degree demand.

Upper bounds on this parameter were found by J.H. Hattingh and M.A. Henning in [2] and by the current author in [5]. Whereas for the strong domination number there are many structural restrictions on the graph (see [2] and [5]) which affect this parameter, similar assumptions have no effect on the weak domination number. Hence, it makes more sense to treat these two parameters separately.

Our first lemma corresponds to Lemma 1 in [2] which gave an analogous bound on $\gamma_{st}(G)$. It can be proved in total analogy to the mentioned Lemma 1.

Lemma 1. Let G be a graph on n vertices. Then $\gamma_w(G) \leq \frac{n+|W(G)|}{2}$.

If we assume a relation between the cardinalities of $W(G)$ and $N(W(G))$, then we can eliminate $|W(G)|$ in Lemma 1.

Lemma 2. Let G be a graph on n vertices, $W = W(G)$ and $|N(W)| \geq \frac{1}{\Delta-1}|W|$ for some $\Delta \geq 2$.

1. $\gamma_w(G) \leq \frac{\Delta n}{\Delta+1}$.
2. $\gamma_w(G) = \frac{\Delta n}{\Delta+1}$ holds if and only if G has the following structure.
   (a) $|V - N[W]| = |N(W)| = \frac{1}{\Delta-1}|W|$.
   (b) $N(V - N[W]) \subseteq N(W)$, i.e. $V(G) - N[W]$ is an independent set.

(c) Every vertex $x \in V(G) - N[W]$ has exactly one weak neighbour in $N(W)$ and every vertex $u \in N(W)$ is a weak neighbour of exactly one vertex in $V(G) - N[W]$.

Proof Claim 1.: Since $V(G) - N(W)$ is a weak dominating set, we obtain $\gamma_w(G) \leq n - |N(W)| \leq n - \frac{1}{\Delta-1}|W|$ which implies to $|W| \leq (\Delta - 1)(n - \gamma_w(G))$. Together with Lemma 1 this yields

$$\gamma_w(G) \leq \frac{\Delta}{2}n - \frac{\Delta - 1}{2}\gamma_w(G)$$

which is equivalent to

$$\gamma_w(G) \leq \frac{\Delta n}{\Delta + 1}.$$

Claim 2.: If $\gamma_w(G) = \frac{\Delta n}{\Delta+1}$, then the proof of the first claim and Lemma 1 imply $\gamma_w(G) = \frac{\Delta}{2}n - \frac{\Delta-1}{2}\gamma_w(G)$ and $|W| = (\Delta - 1)(n - \gamma_w(G))$. This last equality is equivalent to $\gamma_w(G) = n - \frac{1}{\Delta-1}|W|$, which implies $|N(W)| = \frac{1}{\Delta-1}|W|$. Now

$$\gamma_w(G) = \frac{n + |W|}{2} = \frac{\Delta n}{\Delta + 1}$$

implies $n = \frac{\Delta+1}{\Delta-1}|W| = (1 + \frac{2}{\Delta-1})|W|$ and $\gamma_w(G) = \frac{\Delta}{\Delta-1}|W| = (1 + \frac{1}{\Delta-1})|W|$. We obtain for the vertex set $X := V(G) - N[W]$ that $|X| = n - |N(W)| - |W| = (1 + \frac{2}{\Delta-1} - \frac{1}{\Delta-1} - 1)|W| = \frac{1}{\Delta-1}|W|$. Hence, we get part (a).

By the definition, every vertex in X has at least one weak neighbour.

If two vertices $x_1, x_2 \in X$ are joined by an edge in G (we may assume $d(x_1) \leq d(x_2)$), then the set $W \cup X - \{x_2\}$ would be a weak dominating set of G of cardinality $\frac{\Delta}{\Delta-1}|W| - 1$ which is a contradiction to the equations above. Hence, $N(X) \subseteq N(W)$ which implies part (b).

If two vertices $x_1, x_2 \in X$ have a common weak neighbour $u \in N(W)$, then the set $W \cup X \cup \{u\} - \{x_1, x_2\}$ would be a weak dominating set of G of cardinality $\frac{\Delta}{\Delta-1}|W| - 1$ which is once again a contradiction.

Hence, every vertex in $N(W)$ is a weak neighbour of at most one vertex in X. Together with $|X| = |N(W)|$ this implies part (c).

If we assume that G satisfies (a),(b) and (c), then $\gamma_w(G) = \frac{\Delta n}{\Delta+1}$ is immediate. ∎

3. Sharp upper bounds on $\gamma_w(G)$

Our first upper bound on the weak domination number of a graph G will depend on the maximum degree $\Delta(G)$ of G.

Theorem 1. Let G be a connected graph on $n \geq 2$ vertices and of maximum degree $\Delta = \Delta(G)$. Then

$$\gamma_w(G) \leq \frac{\Delta n}{\Delta + 1},$$

where equality holds if and only if G is a star ($G = K_{1,\Delta}$).

Proof If $\Delta = 1$ or $W = W(G) = \emptyset$, the theorem is trivial. Thus, we assume $\Delta \geq 2$ and $W \neq \emptyset$. We consider two cases.

Case 1.: $d(w) = 1$ for all vertices $w \in W$.

Let G_1 be the subgraph of G containing the vertices in W and $N(W)$ and all edges of G joining a vertex in W to a vertex in $N(W)$. Then all connected components of G_1 are stars. If $G_1 = K_{1,n-1}$, then $G = G_1$ and the theorem holds. Thus, we assume $G_1 \neq K_{1,n-1}$. Since G is connected, we obtain that $\Delta(G_1) \leq \Delta - 1$ and all components of G_1 are stars $K_{1,l}$ with $l \leq \Delta - 1$. This yields $|N(W)| \geq \frac{1}{\Delta-1}|W|$. Together with Claim 1 of Lemma 2 this yields the desired inequality.

Now suppose $\gamma_w(G) = \frac{\Delta n}{\Delta+1}$. Then G satisfies the properties (a),(b) and (c) given in Claim 2 of Lemma 2.

Property (a) implies that all components of G_1 are stars $K_{1,\Delta-1}$. Since G is connected, all vertices $u \in N(W)$ have at least one neighbour in $V(G) - W$. Thus, $d(u) = \Delta$ for all $u \in N(W)$ and all these vertices have exactly one such neighbour.

Properties (b) and (c) imply now that all vertices $u \in N(W)$ have exactly one neighbour in $V(G) - N[W]$ which implies the contradiction $d(x) = 1$ for all vertices $x \in V(G) - N[W]$.

Hence, for $G_1 \neq K_{1,n-1}$ we cannot have equality in the given bound.

Case 2.: Not all vertices in W have degree one.

Let G_2 be the graph which arises from G by replacing each vertex $w \in W$ by $d(w)$ vertices $w_1, ..., w_{d(w)}$ all having degree one in G_2 so that $N(w) = \cup_{i=1}^{d(w)} N(w_i)$. We have $n(G_2) - n = k \geq 1$, $\Delta(G_2) = \Delta$ and G_2 satisfies the condition of Case 1.

For every weak dominating set D_2 of G_2 and every $w \in W$ we have $w_i \in D_2$ for every $i = 1, ..., d(w)$. The set $D = D_2 \cup W - \cup_{w \in W}\{w_1, ..., w_{d(w)}\}$ is a weak dominating set of G with $|D| = |D_2| - k$. Hence, we obtain $\gamma_w(G) \leq \gamma_w(G_2) - k \leq \frac{\Delta}{\Delta+1}(n+k) - k < \frac{\Delta n}{\Delta+1}$. This completes the proof. ■

The extremal graphs in Theorem 1 have minimum degree 1. Thus by demanding a higher minimum degree we can obtain the following better upper bound depending on both the maximum and minimum degree.

Corollary 1 . Let G be a connected graph on $n \geq 2$ vertices of maximum degree $\Delta = \Delta(G)$ and minimum degree $\delta = \delta(G)$. Then for $W = W(G)$

$$\gamma_w(G) \leq \frac{\Delta n}{\Delta + 1} - \frac{\delta - 1}{\Delta + 1}|W|. \tag{1}$$

If $\delta \neq \Delta$, then equality holds in (1) if and only if G is a bipartite graph with partite sets W and $N(W)$ and $d(w) = \delta$ for all $w \in W$ and $d(u) = \Delta$ for all $u \in N(W)$. If $\delta = \Delta$, then equality holds in (1) if and only if $G = K_2$, i.e. the complete graph on 2 vertices.

Proof For $\delta = 1$ this is exactly Theorem 1. Hence, we assume $\delta \geq 2$. If $W = \emptyset$, then Lemma 1 implies $\gamma_w(G) \leq \frac{n}{2}$ which implies (1) and equality in (1) can not hold. Hence, we assume $W \neq \emptyset$ which implies $\delta \neq \Delta$. Let G_1 be the graph which arises from G as G_2 in the proof of Theorem 1. We obtain $n(G_1) - n = \sum_{w \in W}(d(w) - 1) \geq$

$(\delta-1)|W|$ where we have $n(G_1)-n=(\delta-1)|W|$ if and only if all vertices $w \in W$ have degree δ. As in the proof of Theorem 1, every weak dominating set D_1 of G_1 yields a weak dominating set D of G with $|D|=|D_1|-(n(G_1)-n)$. Hence, Theorem 1 implies

$$\gamma_w(G) \le \gamma_w(G_1)-(n(G_1)-n) \le \frac{\Delta}{\Delta+1}n(G_1)-(n(G_1)-n)$$
$$= \frac{\Delta}{\Delta+1}n - \frac{1}{\Delta+1}(n(G_1)-n) \le \frac{\Delta}{\Delta+1}n - \frac{1}{\Delta+1}(\delta-1)|W|.$$

We have equality in the above relation if and only if $\gamma_w(G_1)=\frac{\Delta}{\Delta+1}n(G_1)$ and $n(G_1)-n=(\delta-1)|W|$. Hence, all components of G_1 are stars $K_{1,\Delta}$ and all vertices $w \in W$ have degree δ. This implies that G has the demanded structure.

Conversely, it is easy to see that for bipartite graph with the mentioned properties we get equality in the given relation. This completes the proof. ■

The disadvantage of Corollary 1 is that the bound contains the cardinality of the set $W(G)$. Our last corollary offers a way of eliminating $|W(G)|$.

Corollary 2. Let G be a connected graph on $n \ge 2$ vertices of maximum degree $\Delta=\Delta(G)$ and minimum degree $\delta=\delta(G)$. Then

$$\gamma_w(G) \le \frac{\Delta+\delta-1}{\Delta+2\delta-1}n.$$

Proof Corollary 1 gives us an upper bound on $|W(G)|$ in terms of n, Δ, δ and $\gamma_w(G)$. If this bound is inserted in Lemma 1 the given bound is obtained. ■

4. Probabilistic upper bound on $\gamma_w(G)$

It is a well known result that

$$\gamma(G) \le \frac{n(G)(1+\ln(\delta(G)+1))}{\delta(G)+1}$$

(see for example [1]). In order to find an analogous result for $\gamma_w(G)$, we have to find a suitable replacement for $\delta(G)$ in the above expression.

Since every weak dominating set D contains $W(G)$ and all vertices in $N(W(G))$ have a weak neighbour in $W(G)$, we consider only the vertices in $V(G)-[W(G)\cup N(W(G))]$ for graphs for which this vertex set is not empty. The proof of the following result is streamlined from [1].

Theorem 2. Let G be a graph on n vertices with

$$V(G)-[W(G)\cup N(W(G))] \ne \emptyset.$$

If $\delta_w = \min\{d_w(v,G) \mid v \in V(G)-[W(G)\cup N(W(G))]\}$, then

$$\gamma_w(G) \le |W(G)| + (n-|W(G)|)\frac{1+\ln(\delta_w+1)}{\delta_w+1}.$$

Proof Set $p = \frac{\ln(\delta_w+1)}{\delta_w+1}$ and choose every vertex from $V(G) - W(G)$ randomly and independently with probability p. The set of all chosen vertices is denoted by S. Let $Y_S \subseteq V^* = V(G) - [W(G) \cup N(W(G))]$ be the set of vertices that neither belong to S nor have a weak neighbour in S. Since all vertices in V^* have at least δ_w weak neighbours in $V(G) - W(G)$, the probability for each vertex $v \in V^*$ to belong to Y_S is at most $(1-p)^{\delta_w+1}$.

The expected value of $|S|$ is $(n - |W(G)|)p$. The expected value of Y_S is at most

$$\begin{aligned}(n - |W(G) \cup N(W(G))|)(1-p)^{\delta_w+1} &\leq (n - |W(G)|)(1-p)^{\delta_w+1} \\ &\leq (n - |W(G)|)e^{-p(\delta_w+1)} = (n - |W(G)|)\frac{1}{\delta_w+1}.\end{aligned}$$

Therefore, the expected value of $|S| + |Y_S|$ is at most

$$(n - |W(G)|)\frac{1 + \ln(\delta_w+1)}{\delta_w+1}.$$

Since $W \cup S \cup Y_S$ is a weak dominating set of G, this immediately implies the desired upper bound. ∎

5. Lower bound on $\gamma_w(G)$

Since $\gamma(G) \leq \gamma_w(G)$ for every graph G, all lower bounds on $\gamma(G)$ are also lower bounds on $\gamma_w(G)$. But as $\gamma(G)$ and $\gamma_w(G)$ can differ considerably, these lower bounds will not be very good in general. The next theorem presents an analog for the well known bound $\gamma(G) \geq \frac{n(G)}{\Delta(G)+1}$.

Theorem 3. Let G be a graph on n vertices with maximum degree Δ and minimum degree δ. Then

$$\gamma_w(G) \geq \max\{|W(G)|, \frac{n + |W(G)|}{\Delta+1}, \frac{n + \Delta - \delta}{\Delta+1}\}.$$

Proof Since $\gamma_w(G) \geq |W|$ for $W = W(G)$ is immediate, we only prove $\gamma_w(G) \geq \frac{n+|W|}{\Delta+1}$ and $\gamma_w(G) \geq \frac{n+\Delta-\delta}{\Delta+1}$. Let D be a minimal weak dominating set. We consider the set E^* of edges which have one endpoint in D and the other in $V - D$. Since every vertex in $V - D$ has a (weak) neighbour in D, we have $|E^*| \geq n - \gamma_w(G)$.

Since $W \subseteq D$ and the vertices in W have degree at most $\Delta - 1$, we have $|E^*| \leq |W|(\Delta - 1) + (\gamma_w(G) - |W|)\Delta$.

Since every weak dominating set contains at least one vertex of minimum degree, we have $|E^*| \leq \delta + (\gamma_w(G) - 1)\Delta$.

Combining these two upper bounds on $|E^*|$ with the lower bound given above we conclude the two lower bounds on $\gamma_w(G)$. ∎

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