Defining sets of G-designs

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Abstract

Several results, analogous to those already obtained for defining sets of $t-(v, k, \lambda)$ designs, are presented in the case of G-designs. Computational methods and trade structures are used to construct minimal defining sets of each possible size for each of the eight non-isomorphic 4-cycle systems of order 9, and for each of the two non-isomorphic 2-perfect 5-cycle systems of order 11. A recursive method of constructing minimal defining sets of infinite classes of m-cycle systems, when $m \equiv 0 \pmod 4$, is also given.

1. Introduction

In 1990, K. Gray [7] defined a *defining set* in a t-(v, k, λ) design to be a subset $\mathcal{S}$ of the blocks of the design that is a subset of no other t-(v, k, λ) design. A defining set of a t-(v, k, λ) design is *minimal* if it contains no proper subset which is a defining set and is *smallest* if there is no defining set of smaller cardinality for the design.

K. Gray has also found several theoretical results concerning defining sets as well as bounds on the size of a minimal defining set in [7] and in his subsequent papers [8, 9]. These papers, as well as K. Gray and Street [10], Greenhill [11], Sarvate and Seberry [17] and Seberry [18] also present smallest defining sets in several small designs. Greenhill's paper [11] further contains an algorithm for finding smallest defining sets of t-(v, k, λ) designs. Ramsay [16] subsequently improved this algorithm for the case $\lambda = 1$, and found smallest defining sets of all the Steiner triple systems of order 15, thereby continuing work begun by Moran [15]. Minimal defining sets have been found for infinite classes of 2-$(v, 3, 1)$ designs by Gower [4, 5], and B. Gray [6] has found the size of the smallest defining sets for the family of symmetric designs associated with $PG(d, 2)$. For a survey of defining sets of t-designs see Street [20].

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In this paper we present some results concerning defining sets in the case of G-designs. A *G-design* of a graph X is a pair $(V, \mathcal{B})$ where $V = V(X)$ is the vertex set of X and $\mathcal{B}$ is a set of edge disjoint subgraphs of X, all isomorphic to a subgraph G, whose union is X. A G-design of K_n (the complete graph on n vertices) is called a *G-design of order n*. A *partial* G-design of X is a G-design of a subgraph Y of X. We may sometimes not mention the vertex set V of a G-design $(V, \mathcal{B})$ but rather simply refer to the G-design $\mathcal{B}$.

A class of G-designs with which we will be particularly concerned is *m-cycle systems.* If G is an m-cycle then a G-design of a graph X is usually called an *m-cycle system* of X and if $X \cong K_n$ then the design is called an *m-cycle system of order n*, or an *$mCS(n)$*. Examples of 4-cycle systems of order 9 are given in Section 3, Table 1.

2. Preliminaries

In this section we present some results for G-designs which are analogous to those obtained for defining sets of t-(v, k, λ) designs. We also obtain results which apply specifically in the case of m-cycle systems.

Definition 2.1 Let $\mathcal{S}$ be a partial G-design of a graph X. Then $\mathcal{S}$ is a (G, X) *defining set* if there is a unique G-design $\mathcal{B}$ of X with $\mathcal{S} \subseteq \mathcal{B}$. A (G, X) defining set $\mathcal{S}$ is *minimal* if no proper subset of $\mathcal{S}$ is a (G, X) defining set. A *smallest* (G, X) defining set is a (G, X) defining set such that no other (G, X) defining set has smaller cardinality.

Definition 2.2 [1] A *G-trade of volume m* is defined to be a pair $\{T_1, T_2\}$ where each T_i consists of m graphs, pairwise edge-disjoint, all isomorphic to a subgraph G, with the copies of G in T_1 distinct from the copies of G in T_2, and with the union of the edge-sets of the graphs in T_1 being identical to the union of the edge-sets of the graphs in T_2. At times we refer to the single set T_1 as a trade, with the understanding that a set T_2 exists. A G-trade T_1 is *minimal* if no proper subset of T_1 is a G-trade.

Examples of smallest and minimal defining sets of the 4-cycle systems of order 9 are given in Section 3, Table 1. An example of a 5-cycle trade of volume 4 is given in Section 4, Figure 1.

Let $\mathcal{S}$ be a partial G-design of a graph X and let Y be the graph with $V(Y) = V(X)$ and $E(Y) = E(X) \setminus (\cup_{H \in \mathcal{S}} E(H))$. Then $\mathcal{S}$ is a (G, X) defining set if and only if

1. there is a G-design of Y and

2. Y does not contain a G-trade.

The graph $\alpha(H)$, for any graph H and any permutation α of $V(H)$, is defined by $V(\alpha(H)) = \alpha(V(H))$ and $E(\alpha(H)) = \{\alpha(x)\alpha(y) | xy \in E(H)\}$. An *automorphism* of a G-design $(V, \mathcal{B})$ of X is a permutation $\alpha : V \mapsto V$ such that $\alpha(\mathcal{B}) = \mathcal{B}$. The group of all automorphisms of $\mathcal{B}$ is denoted by $\mathrm{Aut}(\mathcal{B})$.

Lemma 2.3 If $\mathcal{S}$ is a (G, X) defining set of the G-design $\mathcal{B}$ and $\alpha \in \mathrm{Aut}(\mathcal{B})$, then $\alpha(\mathcal{S})$ is also a defining set of $\mathcal{B}$ and $\mathrm{Aut}(\mathcal{S}) \subseteq \mathrm{Aut}(\mathcal{B})$.

Proof Clearly, since $\mathcal{S}$ is a defining set of $\mathcal{B}$, $\alpha(\mathcal{S})$ is a defining set of $\alpha(\mathcal{B}) = \mathcal{B}$. Moreover, if $\beta \in \mathrm{Aut}(\mathcal{S})$ then clearly $\beta(\mathcal{S})$ is a subset of both $\mathcal{B}$ and $\beta(\mathcal{B})$, where $\beta(\mathcal{B})$ is some extension of β. But $\beta(\mathcal{B})$ will be a G-decomposition of X and since $\beta(\mathcal{S}) \subseteq \beta(\mathcal{B})$ and $\beta(\mathcal{S})$ is a defining set of $\mathcal{B}$, we must have $\beta(\mathcal{B}) = \mathcal{B}$. Hence $\beta \in \mathrm{Aut}(\mathcal{B})$. □

Definition 2.4 A G-design $\mathcal{B}$ is *single-transposition-free (STF)* if no member of $\mathrm{Aut}(\mathcal{B})$ is a single transposition.

Lemma 2.5 Any defining set $\mathcal{S}$ of an STF G-design $\mathcal{B}$ of order v has at least $v - 1$ distinct vertices occurring in the graphs of $\mathcal{S}$.

Proof If two vertices a and b do not occur in any of the graphs of $\mathcal{S}$ then the transposition $(ab) \in \mathrm{Aut}(\mathcal{S})$. Hence by Lemma 2.3 $(ab) \in \mathrm{Aut}(\mathcal{B})$, a contradiction of the fact that $\mathcal{B}$ is STF. □

Lemma 2.6 Let c_1 and c_2 be edge disjoint m-cycles and let $a, b \in V(c_1) \cap V(c_2)$ such that the distance from a to b in c_1 is equal to the distance from a to b in c_2. Then $T_1 = \{c_1, c_2\}$ is an m-cycle trade.

Proof Let p and q be equal length paths in c_1 and c_2 respectively with common endpoints a and b. Then $T_1 = \{c_1, c_2\}$, $T_2 = \{(c_1 \setminus p) \cup q, (c_2 \setminus q) \cup p\}$ is an m-cycle trade. □

Lemma 2.7 All m-cycle systems are STF.

Proof Let (V, C) be an m-cycle system. If $m = 3$ the m-cycle system is a block design and the result follows from one given in [8]. Suppose $(ab) \in \mathrm{Aut}(\mathcal{C})$. If $m > 3$ then there is a cycle $(a, b, x_3, \ldots, x_m) \in \mathcal{C}$. But this implies that the cycle $(b, a, x_3, \ldots, x_m)$ is also in $\mathcal{C}$; a contradiction. □

Definition 2.8 [12] Given an m-cycle c, let $c(2)$, the *distance* 2 *graph* of c, be the graph formed by joining vertices that are distance 2 apart in c. For example, if c were the cycle (a, b, c, d, e) then the graph $c(2)$ would be the cycle (a, c, e, b, d). Now let $(V, \mathcal{C})$ be an $mCS(n)$ and set $\mathcal{C}(2) = \{c(2) | c \in \mathcal{C}\}$. If $(V, \mathcal{C}(2))$ is also a cycle system of order n, then $(V, \mathcal{C})$ is said to be 2-*perfect*.

3. 4-cycle systems of order 9

It has been established (see [3]) that there are exactly eight non-isomorphic 4-cycle systems of order 9. The following table gives, for each of these systems, a minimal defining set of each possible size. The numbers beside each cycle indicate that it belongs to the given minimal defining set of that size. These results were obtained using computational methods, described fully in [13].

System 1	System 2	System 3	System 4
0123 {4,5,6}	0123 {4,5}	0123 {5}	0123 {5,6}
0245 {5,6}	0245 {5}	0245 {5}	0245 {4,5}
0476 {4,6}	0476 {4,5}	0476 {5}	0476 {5}
0718 {5}	0718	0718	0718 {4}
1346	1346	1346	1364 {5,6}
1485 {4,5}	1485 {4}	1485	1526 {4,6}
2536 {5,6}	2537 {4}	2538 {5}	2758 {5,6}
2738 {4,6}	2638 {5}	2637	3487 {4,6}
5687 {6}	5687 {5}	5687 {5}	3568 {6}
System 5	System 6	System 7	System 8
0123 {4,5}	0123 {4,5}	0123 {4,5}	0123 {5}
0245 {5}	0245 {5}	0245 {4,5}	0245 {5}
0476 {5}	0476 {4,5}	0476 {4,5}	0476 {5}
0718 {4}	0728 {4}	0728	0758
1375 {5}	1346 {5}	1357 {5}	1356 {5}
1436 {4}	1487	1468 {5}	1468 {5}
2538 {5}	1568 {4}	1526	1527
2687	2536	3487	2638
4658 {4}	3758 {5}	3658 {4}	3487

Table 1: $4CS(9)$s with minimal defining sets of each possible size

4. 2-Perfect 5-cycle systems of order 11

It has been determined by computer search [13] that there are exactly two non-isomorphic 2-perfect $5CS(11)$s. In a 2-perfect 5-cycle system, any pair of vertices a and b must occur together in a cycle exactly twice, once as adjacent vertices and once at distance 2. Thus any 2-perfect $5CS(n)$ gives rise to a 2-$(v, 5, 2)$ design with $v = n$, by considering the cycles as blocks. The two 2-perfect $5CS(11)$s are cyclic and can be generated using the starter cycles (0,3,4,8,2) (System A) and (0,8,3,2,4) (System B). The first eight cycles of each system give minimal defining sets. In this section we will discuss several properties of these systems and give some results on their defining sets.

For brevity, a 5-cycle trade will be referred to as a trade, and we will assume that the vertex set of the $5CS(11)$'s is $V = \{p, q, r, s, t, u, v, w, x, y, z\}$.

Lemma 4.1 There are no trades of volume 2 in the 2-perfect 5-cycle systems of order 11.

Proof Let $T_1 = \{c_1, c_2\}$ and $T_2 = \{c'_1, c'_2\}$ and suppose that $\{T_1, T_2\}$ is such a trade. Let $c_1 = (p, q, r, s, t)$. Since the 2-perfect $5CS(11)$'s are also 2-(11, 5, 2) designs, every two cycles must intersect in exactly two vertices. So, without loss of generality $c_2 = (p, r, x, y, z)$. It is obvious that there is only one way in which to decompose the graph $c_1 \cup c_2$ into 5-cycles, and the result follows. □

The first three of the following four lemmas have been established by computation [13].

Lemma 4.2 There are no trades of volume 3 in System A. □

Lemma 4.3 Every set of four cycles in System A is a trade. □

Lemma 4.4 Every pair of cycles in System B is in exactly three trades of volume 3.

Proof Recall that every pair of cycles intersect in two vertices, and consider any two cycles, (p, x, q, y, z) and (p, q, u, v, w). There are three cycles which contain neither p nor q, and it has been verified computationally that those three cycles have the form $(u, w, x, z, *)$, $(v, z, w, y, *)$ and $(v, y, x, u, *)$ (where $*$ is any allowed element). In each case we obtain a trade of volume 3:

1. $T_1 = \{pxqyz, pquvw, uwxz*\}$ and the permutation (px) generates T_2;
2. $T_1 = \{pxqyz, pquvw, vzwy*\}$ and the permutation (zw) generates T_2;
3. $T_1 = \{pxqyz, pquvw, vyxu*\}$ and the permutation (xq) generates T_2;

The fact that these are the only trades of volume 3 has been verified by computer. Hence every pair of cycles is in exactly three trades of volume 3. □

Since every pair of cycles in System B is in three trades of volume 3, there are $\binom{11}{2} \times 3 \div \binom{3}{2} = 55$ trades of volume 3.

Lemma 4.5 Every set of four cycles in System B is a trade of volume 4, or contains a trade of volume 3.

Proof If the four cycles share a common vertex, then it has been shown using *nauty* [14] that they are isomorphic to T_1 in Figure 1, and hence are a trade of volume 4.

It remains to be shown that any four cycles not sharing a common vertex must contain a trade of volume 3. Call the four cycles c_1, c_2, c_3 and c_4, and assume that

T_1	T_2
08324	08624
19435	19467
2a546	2a543
41768	41538

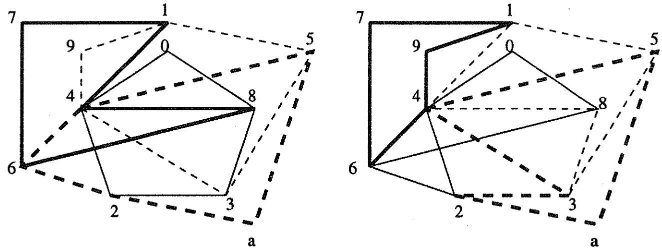

Figure 1: Trade of volume 4

they do not contain a trade of volume 3. Without loss of generality $c_1 = (p, q, r, s, t)$ and $c_2 = (p, r, x, y, z)$. If $p \notin c_3$ and $r \notin c_3$ then by Lemma 4.4 there exists a trade $T_1 = \{c_1, c_2, c_3\}$ and similarly if $p \notin c_4$ and $r \notin c_4$. Also p and r cannot occur together in a cycle again. So, without loss of generality, $c_3 = \{p, b_1, b_2, b_3, b_4\}$ and $c_4 = \{r, b_5, b_6, b_7, b_8\}$ where $b_i \in V \setminus \{p, r\}$.

Now $|c_3 \cap c_1| = |c_3 \cap c_2| = 2$, leaving two vertices of c_3 to be chosen from $\{u, v, w\}$. Similarly c_4 will contain two of $\{u, v, w\}$. Also $|c_3 \cap c_4| = 2$, at least one of which will be from $\{u, v, w\}$. Let $\{c_3 \cap c_4\} = \{d_1, d_2\}$ and let $d_1 \in \{u, v, w\}$. If $d_2 \in \{u, v, w\}$ then by Lemma 4.4, $T_1 = \{c_1, c_3, c_4\}$ is a trade. Similarly if $d_2 \in \{q, s, t\}$ then $T_1 = \{c_2, c_3, c_4\}$ is a trade, and if $d_2 \in \{x, y, z\}$ then $T_1 = \{c_1, c_3, c_4\}$ is a trade. Hence $\{c_1, c_2, c_3, c_4\}$ must contain a trade of volume 3, and the result follows. □

Theorem 4.6 The minimal defining sets of System A are precisely the sets of eight cycles.

Proof Recall that a defining set must contain at least one cycle from each trade. Thus, by Lemma 4.3, a minimal defining set of System A must contain at least eight cycles. Also, since System A has no trades of volumes 2 or 3, a minimal defining set must miss at least three cycles, thereby containing at most eight cycles. Hence, any set of eight cycles from System A form a minimal defining set. □

Since every set of eight cycles is a minimal defining set for System A, there are $\binom{11}{8} = 165$ minimal defining sets. Classification using *nauty* [14], shows that the minimal defining sets fall into three isomorphism classes. Representative defining sets from each of the isomorphism classes are cycles 1,2,3,4,5,6,7,8, cycles 1,2,3,4,5,6,7,9, and cycles 1,2,3,4,5,6,7,10.

Theorem 4.7 A minimal defining set of System B must contain exactly eight cycles.

Proof Clearly, there can be no minimal defining set containing ten cycles. Suppose there is a minimal defining set of nine cycles. Without loss of generality suppose that cycles c_1 and c_2 are not in the defining set. Using Lemma 4.6, there are six cycles for which $\{c_1, c_2, c_i\}$ is not a trade of volume 3. Removing any one of those cycles leaves a defining set, so the original defining set was not minimal; a contradiction. Therefore a minimal defining set of System B must contain at most eight cycles. Furthermore, every set of four cycles is a trade, or contains a trade, so a minimal defining set must contain at least eight cycles. Hence a minimal defining set of System B contains exactly eight cycles. □

For System B not every set of eight cycles will be a minimal defining set, as at least one cycle from each trade of volume 3 must be included in the defining set. Hence there are $\binom{11}{8} - 55 = 110$ minimal defining sets. Classification using *nauty* [14] shows that these minimal defining sets fall into two isomorphism classes. Representative defining sets of each isomorphism class are cycles 1,2,3,4,5,6,7,8 and cycles 1,2,3,4,5,6,7,10.

5. Minimal defining sets of 4x-cycle systems

In this section we show how to construct minimal defining sets of infinite classes of m-cycle systems when $m \equiv 0 \pmod 4$. From an m-cycle system of order n for which a minimal defining set is known, we construct an m-cycle system of order $n + 8x$, where $m = 4x$, and describe a minimal defining set of the new system, see Theorem 5.4. First we need the following lemmas and corollary.

Lemma 5.1 For all integers $x \geq 1$ there exists a $4x$-cycle system of K_{8x+1} which contains a pair of vertex disjoint $4x$-cycles. See for example the first system in table 1.

Proof Let $G_1 \cong K_{4x+1}$ have vertex set $\{0, a_1, a_2, \ldots, a_{4x}\}$ and let $G_2 \cong K_{4x+1}$ have vertex set $\{0, b_1, b_2, \ldots, b_{4x}\}$. It was shown in [2] that for all $n \geq 3$ there exists a decomposition of K_{2n+1} into n $2n$-cycles and an n-cycle. Let C_1 be a $4x$-cycle decomposition of $G_1 \setminus (0, a_1, a_2, \ldots, a_{2x-1})$ and let C_2 be a $4x$-cycle decomposition of $G_2 \setminus (b_1, b_2, \ldots, b_{2x})$. Also, let C_3 be a $4x$-cycle decomposition of $K_{4x,4x}$ (with vertex set $a_1, a_2, \ldots, a_{4x}, b_1, b_2, \ldots, b_{4x}$ and the obvious bipartition); such a decomposition exists, see Sotteau [19]. We can assume that $c = (a_1, b_1, a_2, b_2, \ldots, a_{2x}, b_{2x}) \in C_3$.

Then, $C = C_1 \cup C_2 \cup (C_3 \setminus \{c\}) \cup \{(a_1, a_2, b_1, b_2, a_3, b_3, a_4, b_4, \ldots, b_{2x}), (0, a_1, b_1, b_{2x}, b_{2x-1}, b_{2x-2}, \ldots, b_2, a_2, a_3, a_4, \ldots, a_{2x-1})\}$ is a $4x$-cycle system of K_{8x+1} with vertex set $V(G_1) \cup V(G_2)$. Morever, in C there must be a $4x$-cycle with vertices $a_1, a_2, \ldots, a_{4x}$ and a $4x$-cycle with vertices $0, b_2, b_3, \ldots, b_{4x}$; these two $4x$-cycles are vertex disjoint. □

Corollary 5.2 There exists a $4x$-cycle system C of K_{8x+1} with minimal defining set $\mathcal{S}$ such that there is a pair of vertex disjoint $4x$-cycles in $\mathcal{C} \setminus \mathcal{S}$.

Proof By Lemma 5.1 there exists a $4xCS(8x+1)$, say $\mathcal{C}$, which contains a pair of vertex disjoint $4x$-cycles, say c_1 and c_2. Clearly $\mathcal{C} \setminus \{c_1, c_2\}$ is a defining set of $\mathcal{C}$ and hence some subset of $\mathcal{C} \setminus \{c_1, c_2\}$ is a minimal defining set of $\mathcal{C}$. □

We also need the following lemma, the proof of which uses the construction techniques of [19].

Lemma 5.3 There exists a $4x$-cycle system of $K_{2y,2x}$ $(y \geq x)$, with vertex set $A = \{a_1, a_2, \ldots a_{2y}\} \cup B = \{b_1, b_2, \ldots b_{2x}\}$, such that b_i and b_{i+1}, $i = 1, 2, \ldots (2x-1)$, occur at distance two in every $4x$-cycle.

Proof Such a decomposition is given by the cycles:

$$C_k = (a_{1+2k}, b_1, a_{2+2k}, b_2, \ldots a_{2x+2k}, b_{2x})$$

where $k = 0, 1, \ldots (y-1)$. □

The following theorem shows how to construct minimal defining sets for infinite classes of $4x$-cycle systems.

Theorem 5.4 Let x and n be positive integers with $n > 2x$ and let $\mathcal{M}$ be a minimal defining set for a $4x$-cycle system $\mathcal{D}$ of $G \cong K_n$. Also, let $\mathcal{S}$ be a minimal defining set, as given by Corollary 5.2, of a $4x$-cycle system $\mathcal{C}$ of $H \cong K_{8x+1}$ with $V(H) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup \{a\}$ where

$$\begin{aligned} V_1 &= \{s_1^1, s_2^1, \ldots, s_{2x}^1\}; \\ V_2 &= \{s_1^2, s_2^2, \ldots, s_{2x}^2\}; \\ V_3 &= \{s_1^3, s_2^3, \ldots, s_{2x}^3\}; \\ V_4 &= \{s_1^4, s_2^4, \ldots, s_{2x}^4\}. \end{aligned}$$

Further, $c_1 = (s_1^1, s_1^3, s_2^1, s_2^3, \ldots, s_{2x}^1, s_{2x}^3)$ and $c_2 = (s_1^2, s_1^4, s_2^2, s_2^4, \ldots, s_{2x}^2, s_{2x}^4)$ are in $\mathcal{C} \setminus \mathcal{S}$. Finally, for $i = 1, 2, 3, 4$, let R_i be a $4x$-cycle system, as given by Lemma 5.3, of $K_{n-1,2x}$ with vertex set $(V(G) \setminus \{a\}) \cup V_i$ and the obvious bipartition, such that s_1^i and s_2^i are at distance 2 in every $4x$-cycle. Then $\mathcal{X} = \mathcal{M} \cup \mathcal{S} \cup R_1 \cup R_2 \cup R_3 \cup R_4$ is a minimal defining set (of size $|\mathcal{M}| + |\mathcal{S}| + 2(n-1)$) for the $4x$-cycle system $\mathcal{C} \cup \mathcal{D} \cup R_1 \cup R_2 \cup R_3 \cup R_4$ of K_{n+8x}.

Proof First we need to show that $\mathcal{X}$ is a defining set. Since all the edges with one vertex in $V(G) \setminus \{a\}$ and one vertex in $V(H) \setminus \{a\}$ are in one of the cycles of $\mathcal{X}$, any $4x$-cycle in a completion of $\mathcal{X}$ must have all its vertices in $V(G)$ or all its vertices in $V(H)$. Hence, since $\mathcal{M} \subseteq \mathcal{X}$, $\mathcal{D}$ is contained in any completion of $\mathcal{X}$. Similarly, since $\mathcal{S} \subseteq \mathcal{X}$, $\mathcal{C}$ is contained in any completion of $\mathcal{X}$. Hence $\mathcal{X}$ is a defining set.

Now we show that $\mathcal{X}$ is minimal. Let $c \in \mathcal{X}$ and suppose that $\mathcal{X} \setminus \{c\}$ is a defining set. If $c \in \mathcal{M}$ then (since $\mathcal{M}$ is a minimal defining set of $\mathcal{D}$) there exist at least two distinct $4x$-cycle systems $\mathcal{D}$ and $\mathcal{D}'$ of G with $\mathcal{M}\setminus\{c\}\subseteq\mathcal{D}$ and $\mathcal{M}\setminus\{c\}\subseteq\mathcal{D}'$. Hence $\mathcal{D}\cup\mathcal{C}\cup R_1\cup R_2\cup R_3\cup R_4$ and $\mathcal{D}'\cup\mathcal{C}\cup R_1\cup R_2\cup R_3\cup R_4$ are two distinct $4x$-cycle systems containing $\mathcal{X}\setminus\{c\}$; a contradiction. Hence $c\notin\mathcal{M}$. Similarly, $c\notin\mathcal{S}$. Finally, if $c\in R_i$, then $c=(s_1^i,w_1,s_2^i,w_2,u_3\dots,u_{2x},w_{2x})$ where $w_1,w_2,\dots,w_{2x}\in V(G)\setminus\{a\}$ and $\{s_1^i,s_2^i,u_3,u_4\dots,u_{2x}\}=V_i$. But then $c\cup c_j$, where $j=1$ if i is odd and $j=2$ if i is even, is a $4x$-cycle trade, by Lemma 2.6. Since neither c nor c_j is in $\mathcal{X}\setminus\{c\}$ we have a contradiction. Hence $\mathcal{X}$ is minimal. □

Example Using Theorem 5.4 we can construct a minimal defining set of size 24 for a $4CS(17)$ as follows. We take $x=1$ and $n=9$. Let $\mathcal{D}$ be the $4CS(9)$ obtained from System 2 in Section 3 via the isomorphism $(0,1,2,3,4,5,6,7,8)\mapsto(a,b,c,d,e,f,g,h,i)$ so that $\mathcal{M}=\{(a,b,c,d),(a,e,h,g),(b,e,i,f),(c,f,d,h)\}$ is a minimal defining set of $\mathcal{D}$. Also, let $\mathcal{C}$ be the $4CS(9)$ obtained from System 1 in Section 3 via the isomorphism $(0,1,2,3,4,5,6,7,8)\mapsto(0,1,2,3,a,5,6,7,8)$ so that $\mathcal{S}=\{(0,1,2,3),(0,a,7,6),(1,a,8,5),(2,7,3,8)\}$ is a minimal defining set of $\mathcal{S}$. Then we let $c_1=(0,7,1,8)$, $c_2=(2,5,3,6)$, $V_1=\{0,1\}, V_2=\{2,3\}, V_3=\{7,8\}$ and $V_4=\{5,6\}$. Notice that c_1 and c_2 are vertex disjoint and $c_1,c_2\in\mathcal{C}\setminus\mathcal{S}$. Following the notation of Theorem 5.4, we let

$$\begin{aligned}
R_1 &= \{0,b,1,c),(0,d,1,e),(0,f,1,g),(0,h,1,i)\};\\
R_2 &= \{2,b,3,c),(2,d,3,e),(2,f,3,g),(2,h,3,i)\};\\
R_3 &= \{7,b,8,c),(7,d,8,e),(7,f,8,g),(7,h,8,i)\};\\
R_4 &= \{5,b,6,c),(5,d,6,e),(5,f,6,g),(5,h,6,i)\}.
\end{aligned}$$

Then $\mathcal{X}=\mathcal{M}\cup\mathcal{S}\cup R_1\cup R_2\cup R_3\cup R_4$ is a minimal defining set of size 24 for the $4CS(17)$ $(\{0,1,2,3,5,6,7,8,a,b,c,d,e,f,g,h,i\},\mathcal{C}\cup\mathcal{D}\cup R_1\cup R_2\cup R_3\cup R_4)$.

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