Modified Group Divisible Designs with Block Size 4 and $\lambda > 1$

Ahmed M. Assaf
Department of Mathematics
Central Michigan University
Mt. Pleasant, MI 48859

Abstract: It is shown here that the necessary conditions for the existence of MGD[4, λ m, n] for $\lambda \geq 2$ are sufficient with the exception of MGD[4, 3, 6,23].

1. Introduction

We assume that the reader is familiar with the basic concepts of design theory such as pairwise balanced designs (PBD), group divisible designs (GDD), transversal designs (TD), Latin squares, resolvable designs etc. For the definitions of these combinatorial designs see [3]. We shall adopt the following notation: PBD(v, K, 1) stands for a pairwise balanced design on v points, of index unity, and blocks size from K, if K = {k} the PBD is called balanced incomplete block design, B[v, k, 1]; a (k, λ)-GDD of type $1^a, 2^b, 3^c$... denotes a group divisible design with block size k, index λ, and a groups of size 1, b groups of size 2, etc. A (k, 1)-GDD of type m^k is called a transversal design, TD[k, 1, m].

<u>Definition</u> Modified group divisible design, MGD[k, λ , m, n], is a pair (X,B) where $X = \{ (x_i, y_j) / 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\}$ is a set of order mn and B is a collection of k-subsets of X satisfying the following conditions:

1) every pair of points (x_{i1}, y_{j1}) and (x_{i2}, y_{j2}) of X is contained in exactly λ blocks where $i_1 \neq i_2$ and $j_1 \neq j_2$.
2) the pair of points (x_{i1}, y_{j1}) and (x_{i2}, y_{j2}) with $i_1 = i_2$ or $j_1 = j_2$ is not contained in any block.

The subsets $\{(x_i, y_j)/ 0 \leq i \leq m - 1\}$ where $0 \leq j \leq n - 1$ are called groups and the subsets $\{(x_i, y_j)/ 0 \leq j \leq n - 1\}$ where $0 \leq i \leq m - 1$ are called rows.

Lemma 1.1 [1] The necessary conditions for the existence of MGD[k, λ, m, n] are that m, n $\geq$ k, $\lambda(mn + 1 - m - n) \equiv 0 \pmod{k - 1}$ and $\lambda mn(mn + 1 - m - n) \equiv 0 \pmod{k(k - 1)}$.

In [1] it is proved that the necessary conditions are sufficient when k = 3. However, these conditions are not sufficient when k = 4. A counter example is that MGD[4, 1, 6, 24] does not exist because there do not exist two MOLS of order 6. In the case k = 4 and λ = 1 we have the following:

Lemma 1.2 [2] If m, n $\neq$ 6 then MGD[4, 1, m, n] exists if $(n - 1)(m - 1) \equiv 0 \pmod 3$ with the possible exceptions of (m, n) $\in$ E = {(8,10) (10,15) (1,18) (10,23) (19,11) (19,12) (19,14) (19,15) (19,18) (19,23)}. Furthermore, there exists a MGD[4, 1, 6, n] for n = 7, 10, 19.

The following simple but useful lemma comes from the definiton of MGD.

Lemma 1.3 A MGD[k, λ, m, n] exists iff a MGD[k, λ, n, m] exists.

In this paper we are interested in MGD[4, λ, m, n], $\lambda \geq 2$ and m, n $\geq$ 4. We shall prove the following.

Theorem 1.1 Let $\lambda \geq 2$, m, n $\geq$ 4 be positive integers. Then the necessary conditions for the existence of MGD[4, λ, m, n] are sufficient with the possible exception of (m,n,λ) = (6,23,3).

Finally, we close this section with the following remarks about notations and constructions used in the paper:

1) $H_n = \{h_1, h_2, \ldots, h_n\}$ and $C_n = \{c_1, c_2, \ldots, c_n\}$ are n-sets of points; these points understood to be distinct from any other point in the design being constructed.
2) When the design is not additive, we identify $Z_m \times Z_n$ with Z_{mn}, to avoid a long table of blocks. Furthermore, to find out the permutation one needs to list the blocks and in each step we list the point of H_n which is missing, this list is our permutation.

3) If α is a permutation on H_n then by $\{h_{\alpha(i)}\}$ we mean the elements of H_n under powers of α, e.g. if $\alpha = (h_2h_4h_1h_3)$ then $h_{\alpha(i)} \in (h_2h_4h_1h_3)$.

2. Recursive Constructions

We begin this section with a well known recursive construction, see for example [2].

Lemma 2.1 If there exists a PBD(n, K, λ) and for every $k \in K$ there exists a MGD[r, μ, m, k] then there exists a MGD[r, $\lambda\mu$, m, n].

Proof On n groups of order m construct a PMD(n, K, λ) then on each block of order k, where the points of the block are the groups of order m, we construct a MGD[r, μ, m, k].

The application of the above lemma requires the existence of PBD. As in [2] let $K = \{v: v \in PBD(\{4,5,6,7,9\}, 1)\}$, that is, K is the set of all v's such that there exists a PBD[v, {4,5,6,7,9}, 1]. Then we have the following result.

Lemma 2.2 [2] Let $v \geq 4$ be an integer and $v \notin A = \{8,10,11,12, 14,15, 18, 19,23\}$. Then $v \in K$.

Lemma 2.3 [5] Let $\lambda > 0$ and $v \geq 4$ be positive integers. Then if $\lambda(v - 1) \equiv 0 \pmod 3$ and $\lambda v(v - 1) \equiv 0 \pmod{12}$ then there exists a B[v, 4, λ].

The following lemma is also very useful

Lemma 2.4 Let $v \geq 4$, $v \notin \{6,10,11\}$ be an integer. Then $v \in$ PBD({4,7}, 3).

Proof By Lemma 2.2 if $v \notin A$ then $v \in$ PBD({4,5,6,7,9}, 1). Further, by Lemma 2.3 if $v \equiv 0$ or $1 \pmod 4$ then then $v \in$ PBD({4,7}, 3).
This leaves v = 14,15,18,19,23.
For v = 15 there exists a B[15,7,3] [5].
For v = 14 let $X = Z_{14}$ and let α be the permutation $\alpha = (0\ 1 \ldots 6)(7\ 8 \ldots 13)$ then take the distinct images of the following blocks under powers of α

<0 1 2 3 4 5 6 > (orbit lengh one) <0 1 11 13> <0 7 8 9> <0 7 8 11> <0 1 3 7> <0 2 10 12> <0 3 9 12>.
For v = 18 let X = Z_{18} and let α = (0 1...8)(9 10...17). Then take the distinct images of the following blocks under powers of α
<0 1 3 7 9 13 16> <0 1 3 11> <0 1 5 10> <0 11 12 14>
<0 12 16 17> <0 13 14 16>.
For v = 19 let X = Z_{18} U {a} and let α = (0 1...8)(9 10 ... 17). Then take the distinct images of the following base blocks under powers of α
<0 1 4 9 11 16 a> <0 2 4 10> <0 1 4 13> <0 1 3 9> <0 10 11 14>
<0 11 13 14> <0 12 13 16>.
For v = 23 let X = Z_{23} then take the following blocks mod 23
<0 1 2 4 7 12 16> <0 1 4 17> <0 2 9 15>.

3. MGD with Index Even

In the case λ = 2, 4 the necessary conditions for the existence of MGD[4, λ, m, n] are (m - 1)(n - 1) $\equiv$ 0 (mod 3), m, n $\geq$ 4. In the case λ = 6 the necessary condition is m, n $\geq$ 4.

Lemma 3.1 Let m, n $\geq$ 4, (m - 1)(n - 1) $\equiv$ 0 (mod 3) be positive integers then there exists a MGD[4, λ, m, n] for λ = 2, 4.

Proof We prove the lemma for λ = 2 then λ = 4 is obtained by taking two copies of a MGD[4, 2, m, n].
We first construct a MGD[4, 2, 6, n] for every n, n $\equiv$ 1 (mod 3). But if n $\equiv$ 1 or 4 (mod 12) then n $\in$ PBD({4}, 1) and if n $\equiv$ 7 or 10 (mod 12) n $\neq$ 10, 19 then n $\in$ PBD({4,7}, 1) [4]. Applying Lemma 2.1, we only need to construct a MGD[4, 2, 6, n] for n = 4, 7, 10, 19. For n = 7,10,19 the result is given in Lemma 1.2. For n = 4 let X = Z_{24}. Groups are the integers which are equal modulo 4 in Z_{24} and rows are {i, i+3, i+6, i+9}, i = 0,1,2 together with {j, j+3, j+6, j+9}, j = 12,13,14. Let α be the permutation α = (0 1 ... 11) (12 ... 23). Then the required blocks are the distinct images of the following base blocks under powers of α.
<0 2 7 13> <0 13 14 15> <0 2 19 21> <0 1 15 22> <0 1 18 23>.
For all other values of m, n $\geq$ 4, m, n $\neq$ 6, apply Lemma 2.1 with K = {4}, λ = 2, r = 4, μ = 1 and k = 4. Notice that a MGD[4, 1, m, 4] exists for all m $\geq$ 4, m $\neq$ 6, Lemma 1.2.

Lemma 3.2 Let m, n ≥ 4 be positive integers, then there exists a MGD[4, 6, m, n].

Proof Again we treat the case n = 6 separately. In this case if m ≥ 4, m ≠ 6 then the result follows from Lemma 2.1 with n = 6, K = {4}, $\lambda = 6$, k = r = 4, and $\mu = 1$. So we only need to construct a MGD[4, 6, 6, 6] instead we take two copies of a MGD[4, 3, 6, 6] which can be constructed as follows: $X = Z_{30} \cup H_6$. Let α be the permutation $\alpha = (0\ 1\ \ldots\ 14)(15\ \ldots\ 29)(h_3h_2h_1)\ (h_6h_5h_4)$. Groups are {*i*, *i*+5, *i*+10, ..., *i*+25}, *i* = 0,....,4, together with H_6. Rows are {*i*, *i*+3, ..., *i*+12, $h_{\alpha(i+1)}$} , *i* = 0,1,2 together with {j+15, j+18, ..., j+27, $h_{\alpha(j+4)}$}, j = 0,1,2. Then the blocks are the distinct images of the following base blocks under powers of α.
<0 1 2 h_6> <0 4 17 h_6> <0 7 21 h_6> <2 18 26 h_6> <2 20 24 h_6 >
<15 16 17 h_3> <2 25 29 h_3> <0 19 26 h_3> <0 8 21 h_3> <0 11 18 h_3>
<0 1 17 18> <0 2 16 24> <0 2 26 28> <0 7 19 21> <0 4 23 27>
For all other values of m, n ≥ 4, m, n ≠ 6 notice that m ∈ PBD({4}, 6) and a MGD[4, 1, 4, n] exists for all n ≥ 4, m, n ≠ 6 so apply Lemma 2.1 to get the results.

Corollary 3.1 Let $\lambda > 0$ be an even integer, then the necessary conditions for the existence of a MGD[4, λ, m, n] are sufficient.

Proof Let $\lambda = 6s + t$ where t = 0, 2, 4 then a MGD[4, λ, m, n] is constructed by taking s copies of a MGD[4, λ, m, n] with one copy of a MGD[4, t, m, n].

4 MGD with Index Odd

In this section first we treat the cases $\lambda = 3, 5$. By Lemma 1.1 the necessary condition for the case $\lambda = 3$ is m, n ≥ 4. Again we treat the case m = 6 separately.

Lemma 4.1 There exists a MGD[4, 3, 6, n] for all integers n ≥ 4 with the possible exception of (m,n) = (6,23).

Proof For n ≥ 4, n ∉ {8,10,11,12,14,15,18,19,23} then by Lemma 2.2 there exist a PBD (n, {4,5,6,7,9} ,1). Apply Lemma 2.1 we only need to construct a MGD[4, 3, 6, n] for n $\in$ {4,5,6,7,8,9,10,11,12, 14,15,18,19,23}

For n = 4 let X = Z_{20} U H_4. Let α be the permutation α = (0 1 ... 4) (5 6 ... 9)(10 11 ... 14)(15 16 ... 19). Groups are integers which are equal modulo 5 in Z_{20} togother with H_4. Rows are {0, 1,..., 4, h_1} U {5, 6, ..., 9, h_2} U {10, 11,.... 14, h_3} U {15, 16,..., 19, h_4}. Then the blocks are:

1) On Z_{20} construct a MGD[4, 1, 4, 5].

2) Furthermore, take the following blocks under powers of α
< 0 6 12 18> <0 7 14 16> < 0 8 11 h_3> <0 9 13 h_3> <0 6 12 h_3>
<0 8 19 h_2> <0 9 17 h_2> <0 7 19 h_2> <0 13 16 h_1> <0 14 18 h_1>
<0 11 17 h_1> <5 12 19 h_0> <5 13 16 h_0> < 5 14 18 h_0>.

For n = 5 let X = Z_{30}, rows consists of points which are equal modulo 5 and columns consists of points which are equal modulo 6. For blocks take the following base blocks under the action of the group Z_{30}:
<0 1 2 3> <0 2 9 16> <0 3 7 16> <0 3 11 22> <0 4 8 17>.

For n = 6 the results is given in Lemma 3.2.

For n = 7,10,19 the result follows from Lemma 1.2.

For n = 8 let X = $Z_{42} \cup H_6$, and α be the permutation α = (0 ... 41) (h_1 h_2 ... h_6), rows { *i*, *i*+6, ..., *i*+36, $h_{\alpha(i+1)}$}, *i* = 0, ..., 5, groups {j, j+7, ..., j+35} $\cup H_6$, j = 0, ..., 6 and the blocks are the distinct images, of the following base blocks under powers of α
<0 2 25 h_6> <0 3 4 h_6> <24 26 34 h_6> <15 20 31 h_6> <1 21 34 h_6>
<0 1 2 5> <0 3 16 25> <0 4 15 26> <0 5 13 32> <0 8 17 27>.

For n = 9 let X = $Z_{45} \cup H_9$. Let α be the permutation α = (0 ... 44) ($h_1 h_3 h_5 h_7 h_9 h_2 h_4 h_6 h_8$). Rows are {*i*, *i*+9,...,i+36, $h_{\alpha(i+1)}$}, i = 0, ..., 8. Groups are {*i*, *i*+5, ..., *i*+40}, *i* = 0, ..., 4, together with H_9. Then the blocks are the distinct images of the following base blocks under powers of α.
<14 20 31 h_1 > <1 5 8 h_1> <5 12 33 h_1> <1 15 34 h_1>
<1 13 29 h_1 ><4 6 12 h_1> <2 25 26 h_1> <3 16 35 h_1>
<0 1 3 14 > <0 1 4 26> <0 2 8 16> <0 4 11 33>.

For n = 11 let X = Z_{66}, α = (0, ..., 32)(33, ..., 65). Groups are the integers which are equal modulo 11 and rows are {i, i+3, ... , i+30}, i = 0,1,2 together with {j, j+3 ... , j+30}, j = 33,34,35. The blocks are the distinct images of the following base blocks under powers of α.

<0 1 5 65> <0 38 43 57> <0 2 10 61> <0 34 35 39> <0 4 14 45>
<0 36 50 52> <0 13 14 62> <0 37 47 63> <0 2 16 58> <0 40 53 60>
<0 5 34 42> <0 7 46 54> <0 13 56 58> <0 16 35 36> <0 8 49 59>
<0 2 48 65> <0 7 57 61> <0 1 35 37> <0 4 43 53> <0 5 46 59>
<0 7 47 52> <0 8 38 64> <0 10 60 61> <0 13 38 42><0 16 40 48>.

For n = 12 let X = $Z_{66} \cup H_6$. Rows are { *i*, *i*+6, ..., *i*+60 $h_{\alpha(i+1)}$} *i* = 0, ..., 5 and groups { *i*, *i*+11, ..., *i*+55} $\cup H_6$,*i* = 0, ..., 10. Let α be the permutation α = (0 ... 65) ($h_6 h_5 \dots h_1$). Then the blocks are the distinct images of the following base blocks under powers of α.

<0 17 46 h_6> <0 15 50 h_6> <0 34 62 h_6> <3 5 28 h_6> <5 15 62 h_6>
<0 1 3 8> <0 4 20 39> <0 8 21 53> <0 9 23 49> <0 10 25 39>
<0 1 2 5> <0 3 8 49> <0 7 14 35> <0 9 28 43> <0 10 26 39>.

For n = 14 let X = Z_{65} U H_5 U C_{13} U {∞} and α be the permutation α = (0 . . . 64) ($c_1 c_9 c_4 c_{12} c_7 c_2 c_{10} c_5 c_{13} c_8 c_{11} c_6 c_3$) ($h_1\ h_3\ h_5\ h_2\ h_4$). Groups are {i, i+13, ..., i+52, $c_{\alpha(i+1)}$}, i = 0, ...,12 together with H_5 U {∞}. Rows are {i, i+5,..., i+60, $h_{\alpha(i+1)}$}, i = 0,..., 4 together with C_{13} U {∞}. Then the blocks are

I) { <0 33 1 ∞> <33 1 34 ∞> <1 34 2 ∞> <34 2 35 ∞><2 35 3 ∞>} + 5i, i∈ Z_{13}.

II) Take the distinct images of the following base blocks under powers of α:

<8 9 11 c_1> <8 12 19 c_1> <19 25 47 c_1> <11 19 42 c_1>
<18 27 54 c_1>...<10 22 41 c_1> <2 16 40 c_1> <9 25 46 c_1>
<20 37 56 c_1> <5 23 46 c_1> <4 16 c_1 h_1> <4 62 c_1 h_1>
<1 18 c_1 h_1> <3 46 52 h_1> < 3 24 47 h_1> <0 1 3 9> <0 2 18 46>
<0 3 7 11> <0 9 23 47> <0 12 29 43>.

For n = 15,19 the result follows from Lemmas 2.1,2.4 and Lemma 1.2.

For n = 18 let X = Z_{85} U H_5 U C_{17} U {∞} and let the permutation be

$\alpha = (0\ 1\ \ldots\ 84)\ (h_1\ h_4\ h_2\ h_5\ h_3)$
$(c_1 c_8 c_{15} c_5 c_{12} c_2 c_9 c_{16} c_6 c_{13} c_3 c_{10} c_{17} c_7 c_{14} c_4 c_{11})$. Rows are {i, i+5, ... , i+80, $h_{\alpha(i+1)}$ }, i = 0, ... , 4, together with c_{17} U {∞}. Groups are {i, i+17, ... , i+68, $c_{\alpha(i+1)}$}, i = 0, ..., 16, together with H_5 U {∞}.
Blocks are the following:
I) {<0 32 64 ∞> <32 64 11 ∞> <64 11 43 ∞> <11 43 75 ∞>
<43 75 22 ∞>} + 5i, i∈Z_{17}.
II)Take the distinct images of the following base blocks under powers of α
<2 46 c_1 h_1> <2 44 c_1 h_1> <9 13 c_1 h_1> <2 41 33 h_1>
<3 26 59 h_1> <5 23 62 c_1> <1 2 38 c_1> <1 3 64 c_1>
<4 7 56 c_1> <10 12 16 c_1> <8 15 24 c_1> <6 14 43 c_1>
<14 25 52 c_1> <12 24 38 c_1> <3 16 47 c_1> <11 32 54 c_1>
<10 33 57 c_1> <26 45 39 c_1> <15 42 31 c_1> <0 1 19 28>
<0 1 3 7> <0 3 11 29> <0 7 31 43> <0 9 22 66>
<0 12 26 64> <0 14 37 53>.

Theorem 4.1 Let m, n ≥ 4 be integers, then there exists a MGD[4, 3, m, n] with the possible exception of (m,n) = (6,23).

Proof By Lemma 2.4 if m ≠ 6,10,11 then m ∈PBD({4,7}, 3). Apply Lemma 2.1, we only need to construct a MGD[4, 3, m, n] for m ∈ {4,6,7,10,11}, n ≥ 4. The case m = 4 follows from Lemma 1.2 with the exception of MGD[4, 3, 4, 6] which follows from Lemma 4.1. The case m = 7,10 follows from Lemma 1.2 with the possible exceptions of (m,n) = (10,8) (10,15) (10,18) (10,23). But if n = 8,15,18,23 then n ∈ PBD({4,7}, 3), Lemma 2.4. Now apply Lemma 2.1 to get the result. The case m = 6 was treated in Lemma 4.1. The case m = 11, again by Lemma 2.4 and Lemma 2.1 we only need to construct a MGD[4, 3, 11, n] for n = 4,6,7,10,11. For n = 6 see Lemma 4.1 and for n = 4,7,10 see Lemma 1.2.

For n = 11 let X = $Z_{110} \cup H_{11}$ and let let α be the permutation $\alpha = (0\ \ldots\ 54)\ (55\ \ldots\ 109)\ (h_{11}\ h_9\ h_7\ h_5\ h_3\ h_1\ h_{10}\ h_8\ h_6\ h_4\ h_2)$. Rows are {i, *i*+11, ..., *i*+99, $h_{\alpha(i+1)}$ }, *i* = 0, 1, ..., 11. Groups are {i, i+5, ... , i+50} ∪ {j, j+5, ... , j+50} ∪H_{11}, i = 0, ... , 4; j = 55, ..., 59.
Take the distinct images of the following base blocks under powers of α:

<0 4 31 h_{11}> <0 8 26 h_{11}> <0 7 24 h_{11}> <1 10 42 h_{11}>
<55 64 78 h_{11}> <54 63 82 h_{11}> <57 69 98 h_{11}> <59 62 93 h_{11}>
<3 19 99 h_{11}> <5 65 86 h_{11}> <3 24 93 h_{11}> <5 58 75 h_{11}>
<1 20 68 h_{11}> <10 63 81 h_{11}> <5 18 78 h_{11}> <8 57 106 h_{11}>
<10 24 69 h_{11}> <7 56 63 h_{11}> <3 4 95 h_{11}> <1 59 76 h_{11}>
<0 2 3 26> <55 56 74 82> <0 1 18 58> <0 61 63 84>
<0 6 19 56> <0 57 59 95> <0 12 39 57> <0 56 59 63>
<0 2 67 109> < 0 3 89 105> <0 4 76 82> <0 6 68 85>
<0 7 87 103> <0 8 100 101> <0 9 93 106> <0 12 61 62>
<0 1 79 108> <0 2 78 84> <0 4 74 102> <0 6 60 102> <0 7 71 89>
<0 8 83 95> <0 9 81 89> <0 12 75 106> <0 13 85 94>
<0 14 97 101> <0 16 74 86> <0 17 69 85> <0 18 86 109>
<0 19 65 67> <0 21 90 94> <0 23 97 106> <0 26 61 64>
<0 27 91 98> <0 3 65 79>.

Corollary 4.1 Let $\lambda \equiv 3 \pmod 6$ be a positive integer. Then there exists a MGD[4, λ, m, n] for all m, n $\geq$ 4 with the possible exception of (m, n, λ) = (6,23,3).

Proof For a MGD[4, 9, 6, 23], we have shown that 23 $\in$ PBD({4,7}, 3). Now apply Lemma 2.1 with r = 4 and $\mu = \lambda = 3$ to get the result. For all other values of m, n and $\lambda \equiv 3 \pmod 6$ write $\lambda = 6r + 3$ then the blocks of a MGD[4, λ, m, n] are obtained by taking r copies of a MGD[4, 6, m, n] with one copy of a MGD[4, 3, m, n].

The necessary conditions for $\lambda = 5$ are the same as $\lambda = 1$.

Theorem 4.2 Let m, n $\geq$ 4 then a MGD[4, 5, m, n] exists for all $(m - 1)(n - 1) \equiv 0 \pmod 3$.

Proof In this case a MGD[4, 5, m, n] is obtained by taking a MGD[4, 2, m, n] and a MGD[4, 3, m, n].

Corollary 4.2 Let m, n $\geq$ 4 and $\lambda \equiv 1$ or $5 \pmod 6$, $\lambda \geq 2$ be positive integers. Then there exists a MGD[4, λ, m, n] for all $(m -1)(n -1) \equiv 0 \pmod 3$.

5. Result

Combining Corollary 3.1, Corollary 4.1 and Corollary 4.2 gives the proof of Theorem 1.1.

References

[1] A.M. Assaf, Modified group divisible designs, Ars Combinatoria, 29 (1990), 13-20.

[2] A.M. Assaf and R. Wei, Modified group divisible designs with block size 4 and $\lambda = 1$, Discrete Math., submitted.

[3] T. Beth, D. Jungnickel and H. Lenz, Design Theory, Cambridge University Press, 1986.

[4] A. E Brouwer, Optimal Packings of K_{4s} into K_n, J. of Combin. Theory, Ser. A 26 (1979), 278 - 297.

[5] H. Hanai, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975), 255 - 369.

(Received 6/11/96; revised 9/2/97)