Equitable colourings in the Witt designs

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1. Background

There is a vast literature already in existence on colourings in graphs and designs. We refer the interested reader to [**2, 7, 8, 11, 12**]. A major application of such colourings is to sampling and scheduling problems. For an excellent consideration of designs for statistical purposes, see [**13**]; in [**8**], examples of graph colouring applications in scheduling are described.

Let P be a *point set* and B a set of subsets of P which we shall call *blocks*. A *colouring* of (P, B) is a partition of the point set such that no element of B is entirely contained in an element of the partition. A colouring is a *blocking set* if it is a partition into precisely two classes.

A colouring is *equitable* if the partition classes are of at most two consecutive sizes.

Equitable colourings of Steiner triple systems have been studied by Colbourn, Linek and Rosa [**9**] and by Haddad [**10**].

For the five Witt systems based on the Mathieu groups, a thorough analysis of blocking sets has been done by Berardi, Eugeni and Ferri [**3, 4, 5, 6**]. In this paper we study equitable colourings of these designs.

Let $S = S(t, k, v)$ be a design. If S has an equitable colouring in which the s elements of the partition have the same size a, we refer to this as an a^s-*colouring.* If s elements of the partition have size a, and t have size b, where $s, t \geq 1$, we refer to this as an $a^s b^t$-*colouring.* For convenience in what follows, an $a^s b^t$-colouring with $t = 0$ is an a^s-colouring.

We prove the following in section 2:

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Theorem 1. *Let $S = S(t, k, v)$ be a Witt design, and let $v = sa + t(a+1)$ where $s \geq 1$, $t \geq 0$. Then, except for $v = 12$, $a = 6$, $s = 2$, there exists an $a^s(a+1)^t$-colouring of S.*

We abbreviate the Witt systems as S_{11}, S_{12}, S_{22}, S_{23}, S_{24}.

In section 3, we investigate more closely the non-trivial colourings in the large Witt systems, obtaining the following results:

Theorem 2. *In S_{22} and S_{24}, every set of points not on a block is contained in a blocking set. In S_{23}, every set of at most seven points not on a block is contained in a blocking set; this is false in general for sets of larger size.*

Theorem 3. *In S_{22}, every set of at most eight points, not on a block, is contained in a colour class of a $7^2 8^1$-colouring. In S_{23}, every set of at most eight points, not on a block, is contained in a colour class of a $7^1 8^2$-colouring. In S_{24}, every set of at most eight points, not on a block, is contained in a colour class of an 8^3-colouring.*

We wish to thank Don Kreher for providing us with the tables for S_{23}. At the end of the paper, we point out the discrepancies in the table given in [**4**].

2. Theorem 1

We rely heavily on the results of Berardi, Eugeni and Ferri, and will quote precisely the details we need in Results 1, 2, 3, 4 below.

Result 1. [**6**] S_{11} contains no blocking sets. In S_{12}, the blocking sets are precisely the 6-sets which are not blocks.

Result 2. [**3**] In S_{22}, blocking sets have precisely the sizes 7, 8, 9, 10, 11, 12, 13, 14, 15. Those on seven points are the *Fano sets*, that is, the 7-sets which meet each block in one or three points. Those on eight points are the Fano sets union one additional point. There are three types of blocking sets on eleven points.

Result 3. [**4**] In S_{23}, blocking sets have either eleven or twelve points. There are three types of blocking sets on eleven points; the twelve point blocking sets are the complements of these.

Result 4. [**5, 1**] The blocking sets in S_{24} have eleven, twelve or thirteen points. The eleven point blocking sets have a unique structure, and therefore so do their complements, the thirteen point blocking sets. There are three types of blocking sets on twelve points.

We remark here that in [**5**], the authors show that the Witt designs cannot be partitioned into more than two blocking sets.

The following observation is simple but useful. If $a + 1 < k$, the block size, then S has an $a^s(a+1)^t$-colouring precisely when the diophantine equation $ax + (a+1)y = 0$ has a solution for $x, y \geq 0$. We shall call such colourings *trivial equitable colourings.*

Lemma 1. *In S_{11}, only the trivial equitable colourings exist. In S_{12}, the only non-trivial equitable colouring is a 6^2-colouring.*

Proof. This follows from Result 1. □

Lemma 2. *In S_{22}, the non-trivial equitable colourings are precisely of the type $5^2 6^2$, $7^2 8^1$, 11^2. Every 6-set which is not a block is contained in a $5^2 6^2$-colouring in many ways. Every Fano set is a subset of a class of some $7^2 8^1$-colouring. Moreover, the 11^2-colourings arise solely from blocking sets.*

Proof. Because of Result 2, we need only consider the $5^2 6^2$-colourings.

For the first, take any set of six points not forming a block. In the complement, again take any set of six points not forming a block. (Many such sets exist.) Finally, take the remaining ten points and divide them into two sets of five in any way whatever.

We can easily construct $7^2 8^1$-colourings in the following way. Let one set be a Fano set. Divide the complement of this set in any way into a 7-set and an 8-set. This provides us with a $7^2 8^1$-colouring because of Result 2. □

Lemma 3. *In S_{23}, the non-trivial equitable colourings are precisely of the type $7^1 8^2$ and $11^1 12^1$. The $11^1 12^1$-colourings arise solely from blocking sets. In S_{24}, the non-trivial equitable colourings are precisely of the type 8^3 and 12^2. The 12^2-colourings arise solely from blocking sets.*

Proof. Results 3 and 4 above verify the statements about $11^1 12^1$- and 12^2-colourings. The remainder of the proof follows from the lemmas below. □

We recall some of the properties of the designs S_{22}, S_{23}, S_{24} before proceeding with the construction of the colourings.

The system S_{24} is an *extension* of S_{23} and S_{23} a *contraction* of S_{24}. Similarly for S_{23} and S_{22} and for S_{22} and $PG(2, 4)$. Thus, we can consider the point sets of each of these as a subset or a superset of points of the other systems. Following Todd [**14**], we let the point set of S_{22} be $\{1, 2, ..., 22\}$, that of S_{23} be $\{0, 1, 2, ..., 22\}$ and that of S_{24} be $\{0, 1, 2, ..., 22, \infty\}$.

We shall repeatedly use, without necessarily referring to them, the facts that:

- blocks in S_{22} intersect in 0 or 2 points;
- blocks in S_{23} intersect in 1 or 3 points;
- blocks in S_{24} intersect in 0, 2 or 4 points.

Lemma 4. *There exist $7^1 8^2$-colourings in S_{23} such that, when treated as an extension of S_{22}, one of the colour classes contains a Fano set of the S_{22}.*

Proof. We consider the blocks of S_{22} as 'blocks' of S_{23} by identifying each block B of S_{22} with the corresponding block $B \cup \{0\}$ of S_{23}. This produces a map *onto* the blocks of S_{23} which are on $\{0\}$.

Let F be a Fano set in S_{22}. In S_{22}, all blocks meet each other in 0 or 2 points and meet F in 1 or 3 points. Take blocks B and B' of S_{22} which do not intersect, and such that $|B \cap F| = 3$ and $|B' \cap F| = 1$. (These are seen to exist by checking, for instance, the tables given by Berardi in [**3**].) Now by [**4**], the set $(B \cup B') \setminus \{0\} = X$ is a blocking set on 12 points in S_{23}, and its complement Y is a blocking set on 11 points. It follows that neither $X \setminus F$ nor $Y \setminus F$, each sets of 8 points, contains a block of S_{23}. These two sets, along with F, provide a $7^1 8^2$-colouring of S_{23}. □

Lemma 5. *There exist 8^3-colourings in S_{24} such that, when treated as a double extension of S_{22}, one of the colour classes contains a Fano set of the S_{22}.*

Proof. The construction is identical to that of the previous proof with the addition of 0 *and* ∞ to B and B'. Hence, let B and B' be disjoint blocks of S_{22} meeting the Fano set F : $|B \cap F| = 3$, $|B' \cap F| = 1$. In S_{24}, then, $B \cap B' = \{0, \infty\}$ and $(B \cup B') \setminus \{0, \infty\}$ is a blocking set on 12 points by [**5**]. The sets $F \cup \{0\}$, $(B \cup B') \setminus F$ and the remaining eight points form an 8^3-colouring *unless* $F \cup \{0\}$ is a block of S_{24}, in which case use $F \cup \{\infty\}$ instead. □

It is not difficult to construct $7^2 8^1$-, $7^1 8^2$- or 8^3-colourings which are *not* of this type, using the tables in [**3, 4, 14**]. A complete characterization seems difficult. We can, however, prove a partial converse to the above, in that we can show that an 8^3-colouring in S_{24} induces a $7^1 8^2$-colouring in S_{23}, and so on.

Lemma 6. *An 8^3-colouring of S_{24} induces a $7^1 8^2$-colouring in S_{23}; a $7^1 8^2$-colouring of S_{23} induces a $7^2 8^1$-colouring of S_{22}; a $7^2 8^1$-colouring of S_{22} induces a 7^3-colouring of $PG(2, 4)$.*

Proof. Consider an 8^3-colouring of S_{24}. It suffices to find an element x in S_{24} such that for every block B on x, $B \setminus \{x\}$ meets at least two of the colour classes.

Suppose there is no such element x. Let the colour classes be A_1, A_2, A_3. Then there are two points, x_1, x_2, of A_1 such that there exist blocks B_i on x_i with $B_i \setminus \{x_i\} \subseteq A_2$, say, $i = 1, 2$. Since $k = 8$, $t = 5$ and $|A_2| = 8$, this is not possible.

For S_{23}, exactly the same argument can be made.

For S_{22}, there exist points x_1 and x_2 as above in the 8-set, leading to a similar contradiction. □

We note that in the above proof, at most six points of S_{24} or of S_{23} are not suitable as choices of x; at most sixteen points of S_{22} are not suitable as choices for x.

3. The $7^i 8^j$- and $11^i 12^j$-colourings

We begin with the proof of Theorem 2 which indicates that, despite the close connection between S_{22}, S_{23} and S_{24}, there are differences in terms of blocking set structure.

Theorem 2. *In S_{22} and S_{24}, every set of points not on a block is contained in a blocking set. In S_{23}, every set of at most seven points not on a block is contained in a blocking set; this is false in general for sets of larger size.*

Proof. In S_{22} or S_{24}, let A be a set of points on no block. Suppose A^c contains a block B. Let $x \in B$. Any block meeting B at x meets B, and hence A^c, in a second point. (This is false in S_{23}.) Hence $A \cup \{x\}$ still contains no block. We repeat this until we obtain a set $A^* \supseteq A$ such that neither A^* nor its complement contain a block. It follows that A^* is a blocking set.

We now turn to S_{23}, and, without loss of generality, assume that A is a set of seven points on no block. Let $A = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3\}$. Let $B = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ be the unique block on a_1, a_2, a_3, a_4. Let $B' = \{b_1, b_2,$

$b_3, b_4, b_5, b_6, a_7\}$ be the unique block on b_1, b_2, b_3, a_7. Then $|B \cap B'| = 1$ or 3. If $|B \cap B'| = 1$, then by [**4**, §4], $(B \cup B') \setminus (B \cap B')$ is a blocking set containing A. If $|B \cap B'| = 3$, then $A \neq (B \cup B') \setminus (B \cap B')$. Hence B say, contains a point $u \notin A \cup B'$. If B' contains a point $v \notin A \cup B$, then by Theorem 4.2 of [**4**], $(B \cup B') \setminus \{u, v\} \cup \{a, b\}$ contains A and is contained in an eleven point blocking set. If B' contains no such point v, then a_1, a_2, a_3 or a_4 is in $B' \setminus B$ which is a contradiction.

It remains to consider sets of larger size in S_{23}. Here, we need simply note that the set $\{3, 4, 5, 13, 14, 16, 17, 22\}$ is not contained in any blocking set of S_{23}. (The listing of S_{23} as in [**4**] is used, with the corrections noted at the end of this paper.) □

Corollary. *In S_{22}, respectively S_{24}, any set of size greater than fifteen, respectively fourteen, contains a block.*

Theorem 3. *In S_{22}, every set of at most eight points, not on a block, is contained in a colour class of a $7^2 8^1$-colouring. In S_{23}, every set of at most eight points, not on a block, is contained in a colour class of a $7^1 8^2$-colouring. In S_{24}, every set of at most eight points, not on a block, is contained in a colour class of an 8^3-colouring.*

Proof. We begin with S_{24}. Let A_1 be an 8-set (a set of eight points). Then A_1 is contained in a blocking set X by Theorem 2, on eleven, twelve or thirteen points. Let A_2 be an 8-set on $X \setminus A_1$, not a block, and disjoint from A_1. Let $A_3 = S_{24} \setminus (A_1 \cup A_2)$. Then $\{A_1, A_2, A_3\}$ is an 8^3-colouring.

In S_{23}, we begin with a 7-set A_1. By Theorem 2, A_1 is contained in a blocking set X on eleven or twelve points. Choose a 7-set, not a block, on $X \setminus A_1$ and disjoint from A_1. Let this set be Z. We claim that Z is in some 8-set, disjoint from A_1, which is on no block. Suppose, to the contrary, that for any choice of $x \neq y$ in $S_{23} \setminus (X \cup Z)$, the sets $Z \cup \{x\}$ and $Z \cup \{y\}$ both contain blocks. Then these blocks must intersect in at least five points, which is a contradiction since they must be distinct.

Thus, let A_2 be an 8-set on $X \setminus A_1$, not on a block, and disjoint from A_1. Let $A_3 = S_{23} \setminus (A_1 \cup A_2)$. Then $\{A_1, A_2, A_3\}$ is a $7^1 8^2$-colouring.

Now suppose that A_1 is an 8-set not on a block in S_{23}. We cannot apply Theorem 2 and so must devise a different proof. $S_{23} \setminus A_1$ is not a blocking set and so must contain a block B, say. Write $B = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$. We shall build sets A_2 and A_3, putting $b_1, b_2, b_3, b_4 \in A_2$ and $b_5, b_6, b_7 \in A_3$. The points b_5, b_6, b_7 are together on precisely five blocks [4]. Let B_1, B_2, B_3, B_4 be those blocks distinct from B. We now choose $x_i \in B_i \setminus \{b_5, b_6, b_7\}$ subject to the condition that the points x_1, x_2, x_3, x_4 form no block with any 3-subset of $\{b_1, b_2, b_3, b_4\}$. (If for some choice of x_1, x_2, x_3, x_4, this does happen, replace x_1 by $y_1 \in B_1 \setminus \{x_1, b_5, b_6, b_7\}$. By using block intersection sizes, the set $B' = \{y_1, x_2, x_3, x_4, b_5, b_6, b_7\}$ cannot be a block; nor can the set y_1, x_2, x_3, x_4 form a block with any 3-subset of $\{b_1, b_2, b_3, b_4\}$.) Let $A_2 = \{b_1, b_2, b_3, b_4, x_1, x_2, x_3, x_4\}$, $A_3 = S_{23} \setminus (A_1 \cup A_2)$. Clearly, A_3 is not a block. Moreover, because of the choices of x_1, x_2, x_3, x_4, A_2 contains no block. It follows that $\{A_1, A_2, A_3\}$ is a $7^1 8^2$-colouring.

Finally, consider S_{22}. Let A_1 be an 8-set contained in the blocking set X, using Theorem 2. Let Z be a 6-set, not a block, containing $X \setminus A_1$ and disjoint from A_1. The argument used in S_{23} implies that Z is in some 7-set, A_2, disjoint from A_1 and on no block. Let $A_3 = S_{22} \setminus (A_1 \cup A_2)$. Then $\{A_1, A_2, A_3\}$ is a $7^2 8^1$-colouring. □

4. Corrections to the table for S_{23}

With reference to the table given for S_{23} in [**4**], we make the following additions and corrections.

additions:	2	10	11	14	15	17	18
	2	12	13	15	18	20	22
corrections:	0	1	3	7	9	16	18
	0	3	5	8	11	16	22
	0	4	6	8	16	18	19
	1	9	11	12	16	20	22
	1	10	13	19	20	21	22
	1	11	12	14	17	19	21
	2	8	12	16	19	21	22
	5	8	14	15	16	17	19
	5	9	13	16	18	19	22

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