The Binding Number of a Random Graph

Vadim E. Zverovich*

Department II of Mathematics
RWTH Aachen, Aachen 52056, Germany

Abstract

Let $\mathbf{G}$ be a random graph with n labelled vertices in which the edges are chosen independently with a fixed probability p, $0 < p < 1$. In this note we prove that, with the probability tending to 1 as $n \to \infty$, the binding number of a random graph $\mathbf{G}$ satisfies:

(i) $b(\mathbf{G}) = (n-1)/(n-\delta)$, where δ is the minimal degree of $\mathbf{G}$;

(ii) $1/q - \epsilon < b(\mathbf{G}) < 1/q$, where ϵ is any fixed positive number and $q = 1 - p$;

(iii) $b(\mathbf{G})$ is realized on a unique set $X = V(\mathbf{G}) \backslash N(x)$, where $\deg(x) = \delta(\mathbf{G})$, and the induced subgraph $\langle X \rangle$ contains exactly one isolated vertex x.

All graphs will be finite and undirected, without loops or multiple edges. If G is a graph, $V(G)$ denotes the set of vertices in G, and $n = |V(G)|$. We shall denote the neighborhood of a vertex x by $N(x)$. More generally, $N(X) = \bigcup_{x \in X} N(x)$ for $X \subseteq V(G)$. The minimal degree of vertices and the vertex connectivity of G are denoted by $\delta = \delta(G)$ and $\kappa(G)$, respectively. For a set X of vertices, $\langle X \rangle$ denotes the subgraph of G induced by X.

Woodall [5] defined the *binding number* $b(G)$ of a graph G as follows:

$$b(G) = \min_{X \in \mathcal{F}} \frac{|N(X)|}{|X|},$$

where $\mathcal{F} = \{X : \emptyset \neq X \subseteq V(G), N(X) \neq V(G)\}$. We say that $b(G)$ is *realized* on a set X if $X \in \mathcal{F}$ and $b(G) = |N(X)| / |X|$, and the set X is called a *realizing set* for $b(G)$.

Proposition 1 *For any graph G,*

$$\frac{\delta}{n-\delta} \leq b(G) \leq \frac{n-1}{n-\delta}.$$

*On leave from Faculty of Mechanics and Mathematics, Belarus State University, Minsk 220050, Republic of Belarus. Supported by an award from the Alexander von Humboldt Foundation.

Proof. The upper bound is proved by Woodall in [5]. Let us prove the lower bound. Let $X \in \mathcal{F}$ and $\mid N(X) \mid / \mid X \mid = b(G)$, i.e., X is a realizing set. We have $\mid N(X) \mid \geq \delta$, since the set X is not empty. Suppose that $\mid X \mid \geq n - \delta + 1$. Then any vertex of G is adjacent to some vertex of X, i.e. $N(X) = V(G)$, a contradiction. Therefore $\mid X \mid \leq n - \delta$ and $b(G) = \mid N(X) \mid / \mid X \mid \geq \delta/(n-\delta)$. The proof is complete. ∎

Note that the difference between the upper and lower bounds on $b(G)$ in Proposition 1 is less than 1. In the sequel we shall see that the binding number of almost every graph is equal to the upper bound in Proposition 1.

Let $0 < p < 1$ be fixed and put $q = 1 - p$. Denote by $\mathcal{G}(n, \mathbf{P}(edge) = p)$ the discrete probability space consisting of all graphs with n fixed and labelled vertices, in which the probability of each graph with M edges is $p^M q^{N-M}$, where $N = \binom{n}{2}$. Equivalently, the edges of a labelled random graph are chosen independently and with the same probability p. We say that a random graph $\mathbf{G}$ *satisfies a property* Q if

$$\mathbf{P}(\mathbf{G} \text{ has } Q) \to 1 \text{ as } n \to \infty.$$

We shall need the following results.

Theorem 1 (Bollobás [1]) *A random graph* $\mathbf{G}$ *satisfies* $\kappa(\mathbf{G}) = \delta(\mathbf{G})$.

Theorem 2 (Bollobás [1]) *A random graph* $\mathbf{G}$ *satisfies*

$$\mid \delta(\mathbf{G}) - pn + (2pqn \log n)^{1/2} - \Big(\frac{pqn}{8 \log n}\Big)^{1/2} \log\log n \mid \leq C(n) \Big(\frac{n}{\log n}\Big)^{1/2},$$

where $C(n) \to \infty$ *arbitrarily slowly.*

Theorem 3 (Erdös and Wilson [3]) *A random graph has a unique vertex of minimal degree.*

Now we can state the main result of the paper.

Theorem 4 *The binding number of a random graph* $\mathbf{G}$ *satisfies*

$$b(\mathbf{G}) = \frac{n-1}{n-\delta}.$$

Proof. Taking into account Proposition 1, it is sufficient to prove that

$$\frac{\mid N(X) \mid}{\mid X \mid} \geq \frac{n-1}{n-\delta}$$

for any set $X \in \mathcal{F}$. Let $Y = N(X) \backslash X$ and consider three cases.

(i) The induced subgraph $\langle X \rangle$ does not contain an isolated vertex. The set $V(\mathbf{G}) \backslash N(X)$ is not empty, since $X \in \mathcal{F}$. Hence the set Y is a cutset of the graph $\mathbf{G}$. By Theorem 1, $\kappa(\mathbf{G}) = \delta(\mathbf{G})$. Therefore $\mid Y \mid \geq \delta$ and $\mid X \mid < n - \delta$. We have

$$\frac{\mid N(X) \mid}{\mid X \mid} = \frac{\mid Y \mid + \mid X \mid}{\mid X \mid} = \frac{\mid Y \mid}{\mid X \mid} + 1 \geq \frac{n}{n-\delta} > \frac{n-1}{n-\delta}.$$

(ii) The induced subgraph $\langle X \rangle$ contains exactly one isolated vertex. Obviously $| Y | \geq \delta$ and $| X | \leq n - \delta$. Then, taking into account that $\delta(\mathbf{G}) > 0$, we obtain

$$\frac{| N(X) |}{| X |} = \frac{| Y | + | X | - 1}{| X |} = \frac{| Y | - 1}{| X |} + 1 \geq \frac{n-1}{n-\delta}.$$

(iii) The induced subgraph $\langle X \rangle$ contains more than one isolated vertex. If x and y are different vertices of $\mathbf{G}$, then $\deg(x, y)$ denotes the *pair degree* of the vertices x and y, i.e., the cardinality $| N(\{x, y\}) \backslash \{x, y\} |$. Define $\mu = \mu(\mathbf{G}) = \min \deg(x, y)$, where the minimum is taken over all pairs of different vertices $x, y \in V(\mathbf{G})$. Now introduce a random variable ξ on $\mathcal{G}(n, \mathbf{P}(edge) = p)$. The random variable ξ is equal to the number of pairs of different vertices in $\mathbf{G}$ such that

$$\deg(x, y) \leq (1 - q^2 - \epsilon)(n - 2),$$

where ϵ is fixed and $0 < \epsilon < 1 - q^2$. We need to estimate the expectation $\mathbf{E}\xi$. Let the vertices x and y be fixed. Then

$$\Pi = \mathbf{P}(\deg(x, y) \leq k) = \sum_{t \leq k} \binom{n-2}{t} (1 - q^2)^t (q^2)^{n-2-t},$$

where $k = (n - 2)(1 - q^2 - \epsilon)$. We now use the Chernoff formula [2]:

$$\sum_{t \leq k} \binom{m}{t} P^t Q^{m-t} \leq \exp\left(k \log \frac{mP}{k} + (m - k) \log \frac{mQ}{m-k}\right)$$

whenever $k \leq mP$, $P > 0$, $Q > 0$ and $P + Q = 1$. Taking $m = n-2$, $k = m(1-q^2-\epsilon)$, $P = 1 - q^2$ and $Q = q^2$, and noting that $\log x < x - 1$ if $x \neq 1$, we find that

$$\Pi \leq \exp\{(n - 2)\Theta\}$$

where

$$\Theta = (1 - q^2 - \epsilon) \log \frac{1 - q^2}{1 - q^2 - \epsilon} + (q^2 + \epsilon) \log \frac{q^2}{q^2 + \epsilon}$$
$$< (1 - q^2) - (1 - q^2 - \epsilon) + q^2 - (q^2 + \epsilon) = 0.$$

Thus $\Pi < e^{-Cn}$, where $C > 0$ is a constant. At last, we get

$$\mathbf{E}\xi \leq \binom{n}{2} e^{-Cn} = o(1).$$

If ξ is a non-negative random variable with expectation $\mathbf{E}\xi > 0$ and $r > 0$, then from the Markov inequality it follows that

$$\mathbf{P}(\xi \geq r\mathbf{E}\xi) \leq 1/r.$$

Taking $r = 1/\mathbf{E}\xi$, we have $\mathbf{P}(\xi \geq 1) \leq \mathbf{E}\xi = o(1)$, i.e. $\mathbf{P}(\xi = 0) = 1 - o(1)$. Thus

$$\mu > (1 - q^2 - \epsilon)(n - 2).$$

Denote by m the number of isolated vertices in the graph $\langle X\rangle$. Clearly $m \leq \alpha$, where $\alpha = \alpha(\mathbf{G})$ is the independence number of $\mathbf{G}$. It is well-known [4] that for a random graph $\mathbf{G}$, $\alpha(\mathbf{G}) = o(n)$, so that $\mu > \alpha$. Furthermore, $|Y| \geq \mu$ and $|X| \leq n-\mu$, since $m \geq 2$, and so $|Y| - m \geq \mu - \alpha > 0$. We obtain

$$\frac{|N(X)|}{|X|} = \frac{|Y| + |X| - m}{|X|} = \frac{|Y| - m}{|X|} + 1 \geq \frac{\mu - \alpha}{n-\mu} + 1 =$$
$$\frac{n-\alpha}{n-\mu} > \frac{n - o(n)}{n - (1-q^2-\epsilon)(n-2)} = \frac{1}{\epsilon + q^2}(1 - o(1)).$$

On the other hand, by Theorem 2,

$$\frac{n-1}{n-\delta} = \frac{n-1}{n - pn(1-o(1))} = \frac{1}{q}(1-o(1)).$$

Now, if we take $\epsilon < q - q^2$, then we have

$$\frac{|N(X)|}{|X|} > \frac{n-1}{n-\delta}.$$

This completes the proof of Theorem 4. ∎

Using Theorems 2-4, the following corollaries are obtained.

Corollary 1 *If $C(n) \to \infty$ arbitrarily slowly, then the binding number of a random graph $\mathbf{G}$ satisfies*

$$\frac{n-1}{K + C(n)(n/\log n)^{1/2}} \leq b(\mathbf{G}) \leq \frac{n-1}{K - C(n)(n/\log n)^{1/2}},$$

where

$$K = qn + (2pqn\log n)^{1/2} - \left(\frac{pqn}{8\log n}\right)^{1/2} \log\log n.$$

The proof follows immediately from Theorems 2 and 4. ∎

It may be pointed out that the bounds in Corollary 1 are essentially best possible, since the result of Theorem 2 is best possible (see [1]).

Corollary 2 *If $\epsilon > 0$ is fixed, then the binding number of a random graph $\mathbf{G}$ satisfies*

$$1/q - \epsilon < b(\mathbf{G}) < 1/q.$$

The proof follows immediately from Corollary 1. ∎

Corollary 3 *The binding number of a random graph $\mathbf{G}$ is realized on a unique set $X = V(\mathbf{G})\backslash N(x)$, where $\deg(x) = \delta(\mathbf{G})$, and the graph $\langle X\rangle$ contains exactly one isolated vertex x.*

Proof. One may see from the proof of Theorem 4 that the equality

$$|N(X)| / |X| = (n-1)/(n-\delta)$$

for a random graph $\mathbf{G}$ is possible only if the graph $\langle X\rangle$ contains exactly one isolated vertex x and $|X| = n - \delta$. Thus $\deg(x) = \delta(\mathbf{G})$ and $X = V(\mathbf{G})\backslash N(x)$. By Theorem 3, the set X is unique. ∎

Acknowledgment The author thanks the referee for useful suggestions.

References

[1] B. Bollobás, Degree sequences of random graphs, *Discrete Math.* **33** (1981) 1-19.

[2] H. Chernoff, A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Ann. Math. Stat.* **23** (1952) 493-509.

[3] P. Erdös and R.J. Wilson, On the chromatic index of almost all graphs, *J. Combinatorial Theory Ser.* **B 23** (1977) 255-257.

[4] K. Weber, Random graphs - a survey, *Rostock. Math. Kolloq.* **21** (1982) 83-98.

[5] D.R. Woodall, The binding number of a graph and its Anderson number, *J. Combinatorial Theory Ser.* **B 15** (1973) 225-255.

(Received 18/9/96)