On the Flat Antichain Conjecture

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Abstract

We present partial results on the Flat Antichain Conjecture. In particular, we prove that the conjecture is true when the average size of the edges is an integer.

1 Introduction

A hypergraph is a collection of subsets (called *edges*) of a finite set S. If a hypergraph $\mathcal{A}$ is such that $A_i, A_j \in \mathcal{A}$ implies $A_i \not\subseteq A_j$, then $\mathcal{A}$ is called an *antichain*. In other words $\mathcal{A}$ is a collection of pairwise incomparable sets. Antichains are also known under the names *simple hypergraph* or *clutter*. The largest integer less than or equal to a real number x will be denoted by $\lfloor x \rfloor$. The smallest integer greater than or equal to a real number x will be denoted by $\lceil x \rceil$. The set of all k-subsets of an set S will be denoted by $\binom{S}{k}$.

The *Flat Antichain Conjecture*, due to Paulette Lieby [4], was motivated by the study of Completely Separating Systems [7].

Conjecture 1 (Flat Antichain Conjecture) *Given an antichain $\mathcal{A}$ on an n-set S, there exists an antichain $\mathcal{F}$ on S satisfying the following conditions:*

1. $|\mathcal{F}| = |\mathcal{A}|$,
2. $\sum_{E \in \mathcal{F}} |E| = \sum_{E \in \mathcal{A}} |E|$,
3. $\exists k \in [1, n],\ \mathcal{F} \subseteq (\binom{S}{k} \cup \binom{S}{k+1})$.

The first condition says that $\mathcal{A}$ and $\mathcal{F}$ have the same number of edges, the second condition says that their vertices versus edges incidence matrices have the same number of 1's, and the third condition says that $\mathcal{F}$ is *flat*. If $\mathcal{F}$ exists then we say that $\mathcal{A}$ can be *flattened*. The sum $\sum_{E\in\mathcal{A}}|E|$ will be denoted by $t(\mathcal{A})$. We shall prove that the conjecture is true when $t(\mathcal{A})/|\mathcal{A}|$ is an integer. In fact in this case we can be more precise:

Theorem 1 *Let $\mathcal{A}$ be an antichain of an n-set S, such that k=$t(\mathcal{A})/|\mathcal{A}|$ is an integer. Then there exists an antichain $\mathcal{F}$ on S such that*

1. $|\mathcal{F}| = |\mathcal{A}|$,
2. $t(\mathcal{F}) = t(\mathcal{A})$,
3. $\mathcal{F} \subseteq \binom{S}{k}$.

To establish our results we will use the L.Y.M. inequality, which is a generalization of Sperner's theorem (the size of an antichain of an n-set is at most $\binom{n}{\lfloor\frac{n}{2}\rfloor}$). Lubell, Yamamoto and Meschalkin (see [1] or [2] for more details) generalized Sperner's theorem by proving that:

Theorem 2 (The L.Y.M. inequality) *Let p_k denote the number of members of size k of an antichain $\mathcal{A}$. Then*

$$\sum_{k=1}^{n}\frac{p_k}{\binom{n}{k}} \leq 1.$$

In passing, using ideas from [3], we will show that:

Theorem 3 *Let $\mathcal{A}$ be an antichain of an n-set. Then*

$$\sum_{A\in\mathcal{A}}|A| \leq \left\lceil\frac{n}{2}\right\rceil\binom{n}{\lceil\frac{n}{2}\rceil}.$$

In the last section we make some remarks on the general case of the Flat Antichain Conjecture.

2 Proofs

Proof of Theorem 1: To prove that $\mathcal{F}$ exists, it is sufficient to prove that $|\mathcal{A}| \leq \binom{n}{k}$, where k is the integer $t(\mathcal{A})/|\mathcal{A}|$.

Let p_i denote the number of members of size i of the antichain $\mathcal{A}$, and let $m = |\mathcal{A}|$. The L.Y.M. inequality states that:

$$\sum_{i=1}^{n}\frac{p_i}{\binom{n}{i}} \leq 1.$$

Therefore we have:

$$\sum_{i=1}^{n} \frac{p_i}{m} f(i) \leq \frac{1}{m},$$

where $f(i) = 1/\binom{n}{i}$. It is sufficient to prove that $f(k) \leq \frac{1}{m}$. The function f can be extended to the set of reals $[0, n]$: on $[i, i+1]$, with i integer, define for u real in $[0,1]$, $f(i+u) = (1-u)f(i) + uf(i+1)$.

The function f is convex. We shall prove this later. As f is convex and $k = \sum_{i=1}^{n} i\frac{p_i}{m}$, we have

$$f(k) = f(\sum_{i=1}^{n} i\frac{p_i}{m}) \leq \sum_{i=1}^{n} f(i)\frac{p_i}{m} \leq \frac{1}{m}.$$

Hence we have $m \leq \binom{n}{k}$. To complete the proof we still have to show that f is convex. It is sufficient to prove that $f(i) \leq \frac{f(i-1)+f(i+1)}{2}$ for all $i \in \{1, \ldots, n-1\}$. We have

$$\frac{f(i-1)+f(i+1)}{f(i)} = \frac{n-i+1}{i} + \frac{i+1}{n-i} = (\frac{n-i}{i} + \frac{i}{n-i}) + (\frac{1}{i} + \frac{1}{n-i}).$$

But the function $y \to y + 1/y$ is always greater than or equal to 2 on $]0, \infty[$. Therefore f is convex. □

Proof of Theorem 3: We shall translate the hypothesis of Theorem 3 into a linear program. Using the duality theorem of linear programming and the L.Y.M. inequality we will prove the inequality of Theorem 3.

Claim 1

$$\max\{k\binom{n}{k} \mid 1 \leq k \leq n\} = \left\lceil \frac{n}{2} \right\rceil \binom{n}{\lceil \frac{n}{2} \rceil}.$$

Proof of claim: We have

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Therefore, the maximum is obtained when $k - 1 = \left\lfloor \frac{n-1}{2} \right\rfloor$. That is to say when $k = \left\lceil \frac{n}{2} \right\rceil$.

Let A be the $(n+1) \times n$ matrix defined by $A_{1,j} = \binom{n}{j}^{-1}$ if $j \in [1, n]$, $A_{i,i-1} = -1$ if $i \in [2, n+1]$, and $A_{i,j} = 0$ otherwise. Let $c = (1, 2, \ldots, n)$, and $b = (1, 0, 0, \ldots, 0)$. The first row constraint of the system $Ax \leq b$ is the L.Y.M. inequality. (Here the variables in x represent the p_i's.) The other constraints translate the non-negativity of x.

By the duality theorem of linear programming we know that

$$\max\{cx \mid Ax \leq b\} = \min\{yb \mid y \geq 0,\ yA = c\}.$$

Let $y = (y_0, y_1, y_2, \ldots, y_n)$. We have $yb = y_0$ and $\binom{n}{k}^{-1} y_0 - y_k = k$ for $k \in [1, n]$. This can be transformed into $y_0 = \binom{n}{k}(k + y_k)$. The linear program $\min\{yb \mid y \geq 0, yA = c\}$ has the following feasible solution. Take $y_{\lceil \frac{n}{2} \rceil} = 0$. Then $y_0 = \lceil \frac{n}{2} \rceil \binom{n}{\lceil \frac{n}{2} \rceil}$. Because of the claim, it is possible to find suitable values for the other y_i's. Hence we have that

$$\min\{yb \mid y \geq 0, yA = c\} \leq \left\lceil \frac{n}{2} \right\rceil \binom{n}{\lceil \frac{n}{2} \rceil} .$$

Therefore we have also

$$\max\{cx | Ax \leq b\} \leq \left\lceil \frac{n}{2} \right\rceil \binom{n}{\lceil \frac{n}{2} \rceil} .$$

As the parameters $p_1, p_2, \ldots, p_n$ are a feasible solution of $\max\{cx|Ax \leq b\}$, this finishes the proof of theorem 3. □

This bound is best possible since it is achieved by the antichain formed by all the $\lceil \frac{n}{2} \rceil$-sets.

3 Remarks on the General Case of the Conjecture

The profile of a hypergraph $\mathcal{H}$ is the vector $p = (p_0, \ldots, p_n)$, whose entries are

$$p_i = \left| \binom{S}{i} \cap \mathcal{H} \right| .$$

We will show that if the Flat Antichain Conjecture is true, then the profile of the antichain $\mathcal{H}$ is determined by a linear system.

Let p_i denote the number of members of size i of an antichain $\mathcal{A}$. Let $m = |\mathcal{A}|$, and $t = t(\mathcal{A})$. Assume there exists a flat antichain $\mathcal{F}$ such that

1. $|\mathcal{F}| = m$,
2. $t(\mathcal{F}) = t$,
3. $\exists k \in [1, n]$, $\mathcal{F} \subseteq (\binom{S}{k} \cup \binom{S}{k+1})$.

Let $q_k = |\mathcal{F} \cap \binom{S}{k}|$, and let $q_{k+1} = |\mathcal{F} \cap \binom{S}{k+1}|$. Note that q_k, q_{k+1} must satisfy the system

$$\begin{cases} kq_k + (k+1)q_{k+1} = t \\ q_k + q_{k+1} = m. \end{cases}$$

Using the fact that q_k and q_{k+1} are non negative, we deduce that $k = \lfloor \frac{t}{m} \rfloor$. Therefore k is equal to the quotient of the Euclidian division of t by m. More precisely $t = mk + r$ with $0 < r < m$ (the case $r = 0$ is Theorem 1). To sum up, $q_k = m - r$ and $q_{k+1} = r$.

Using a similar argument to that in the proof of Theorem 1 with the equality

$$\frac{t}{m} = k\frac{m-r}{m} + (k+1)\frac{r}{m}$$

we can prove that

$$\frac{q_k}{\binom{n}{k}} + \frac{q_{k+1}}{\binom{n}{k+1}} \leq 1,$$

which is the L.Y.M. inequality. Unfortunalety, this is not a sufficient condition for the existence of an antichain $\mathcal{F}$. There exist families of integers $p_0, \ldots, p_m$ that satisfy the L.Y.M. inequality, but with no antichain having these profiles.

Computer experiments from Paulette Lieby and these partial results make us believe that the conjecture is true.

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References

[1] Anderson I. : Combinatorics of Finite Sets, Oxford science publication, 1987.

[2] Berge C. : Graphs and Hypergraphs, North-Holland, 1985.

[3] Frankl P., Katona G. : Polytopes determined by hypergraph classes, European J. Combin. **6**, 223-243, 1985.

[4] Lieby P. : Honours Thesis (Maths), Northern Territory University, Australia, 1995.

[5] Lubell D. : A short proof of Sperner's theorem, J. Combin. Theory **1**, pp. 299, 1966.

[6] Meschalkin L. D. : A generalization of Sperner's theorem on the number of subsets of a finite set, Theory Probab. Appl. **8**, pp. 203-4, 1963.

[7] Ramsay C., Roberts I.T. : Minimal completely separating systems of sets, Australiasian Journal of Combinatorics, **13**, pp. 129 - 150, 1996.

[8] Sperner E. : Ein Satz uber Untermengen einer endlichen Menge. Math. Z. **27**, pp. 544-8, 1928.

[9] Yamamoto K. : Logarithmic order of free distributive lattices. J . Math. Soc. Japan. **6**, pp. 343-53, 1954.

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