THE MAXIMUM GENUS OF DIAMETER THREE GRAPHS

HUNG-LIN FU[1] AND MING-CHUN TSAI[1]

Department of Applied Mathematics
National Chiao-Tung University
Hsin-Chu, Taiwan, Republic of China

ABSTRACT. The *maximum genus* of a connected graph G, $\gamma_M(G)$, is the largest genus of an orientable surface on which G has a 2-cell embedding, and the *Betti deficiency*, $\xi(G)$, is equal to $\beta(G) - 2\gamma_M(G)$ where $\beta(G) = |E(G)| - |V(G)| + 1$ is the *Betti number* of G.

In this paper we study the maximum genus of a graph with diameter $k \geq 3$ and we prove that the Betti deficiency of a diameter 3 multigraph is at most 2. In the case that the diameter 3 graph G is simple, the Betti deficiency of G can be determined. As to graphs with larger diameter, some partial results are obtained.

§ 1. Introduction

This paper is devoted to an investigation of the maximum genus of graphs. Since the maximum genus is invariant under homeomorphisms, the results we obtain can be extended to graphs which are homeomorphic to the investigated graphs.

Throughout of this paper a graph in which multi-edges and loops are allowed is called a *pseudograph*. A graph without loops is a *multigraph*, and a *simple graph* is a graph which contains no multi-edge and loop. Without mentioning otherwise, a "graph" means a "pseudograph". For basic information and results, the readers may refer to the book *Graphs and Digraphs*[1]. Recall that the maximum genus of a connected graph G, $\gamma_M(G)$, is the largest genus of an orientable surface on which G has a 2-cell embedding, and the Betti deficiency, $\xi(G)$, is equal to $\beta(G) - 2\gamma_M(G)$. Thus the value $\xi(G)$ will naturally determine the maximum genus of G. We note here that $\xi(G) \equiv \beta(G)$ $(mod\ 2)$. Hence $\gamma_M(G)$ can attain its maximum if and only if $\xi(G) = 0$ or 1 (depending on whether $\beta(G)$ is even or odd). In the case that $\xi(G) \leq 1$, the graph G is said to be *upper embeddable*.

There are many good results in the study of the maximum genus[2-12]. Mainly two approaches have been utilized. The first one was proved by Xuong[11].

Theorem 1.1. [11] *Let G be a connected graph. Then $\xi(G) = \min\{\xi(G,T)|\ T$ is a spanning tree of $G\}$ where $\xi(G,T)$ is the number of odd size components in $G - E(T)$.*

The spanning tree T of G which gives $\xi(G,T) \leq 1$ is called a *splitting tree* of G. Therefore the following result is obvious.

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Theorem 1.2. [3,11] *A graph G is upper embeddable if and only if G has a splitting tree.*

In the other direction, Nebeský gave the following result. Let $\nu(G, A) = c(G - A) + b(G - A) - 1 - |A|$, where $c(G - A)$ denotes the number of components in $G - A$ and $b(G - A)$ denotes the number of components in $G - A$ with odd Betti numbers.

Theorem 1.3. [4] *Let G be a connected graph. Then*

$$\xi(G) = \max\{\nu(G, A) | A \subseteq E(G)\}.$$

Thus we have the following corollaries.

Corollary 1.4. *Let G be a connected graph and let k be a positive integer. If $c(G - A) + b(G - A) - k \leq |A|$ for each $A \subseteq E(G)$, then $\xi(G) \leq k - 1$.*

Corollary 1.5. *Let G be a connected graph. Then G is upper embeddable if and only if $c(G - A) + b(G - A) - 2 \leq |A|$ for each $A \subseteq E(G)$.*

From the above results, we derive the following theorems. The first one gives a distinct proof of Škoviera's result[9].

Theorem 1.6. [9] *A diameter 2 multigraph is upper embeddable.*

Theorem 1.7. *Let G be a diameter 3 multigraph. Then $\xi(G) \leq 2$.*

In the case that G is a simple diameter 3 graph, we can determine $\gamma_M(G)$. Before we state the theorem, we introduce two classes of graphs which have Betti deficiency 2.

A graph G is in class S if it contains a bridge v_1v_2 such that

(1) both components G_1 and G_2 of $G - v_1v_2$ which contain v_1 and v_2, respectively, have odd Betti numbers; and
(2) the vertex v_i is adjacent to every vertex in $G_i - v_i$, $i = 1, 2$.

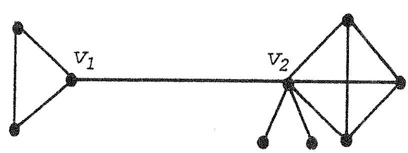

FIGURE 1.1

A graph H is in class T if it contains a 3-element edge subset A such that

(1) $H - A$ consists of three components H_1, H_2 and H_3 with odd Betti numbers; and
(2) if V_i is the set of vertices in H_i incident with the edges in A, then every vertex in V_i is adjacent to all the vertices of $V(H_i) \backslash V_i$, $i = 1, 2, 3$.

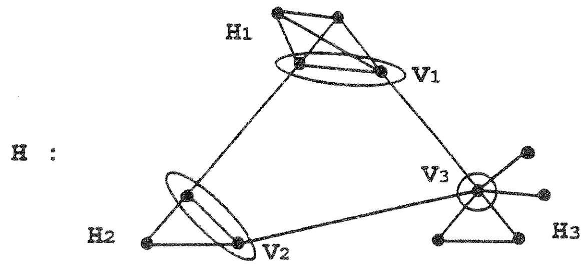

FIGURE 1.2. $A = \{e_1, e_2, e_3\}$

Now we are ready to state the theorem.

Theorem 1.8. *A simple diameter 3 graph G has Betti deficiency 2 if and only if G is in $\mathcal{S}$ or $\mathcal{T}$.*

We note here that if G is a pseudograph with diameter 3, then to determine its maximum genus is going to be very difficult. The graph in Figure 1.3 shows that the Betti deficiency of a diameter 3 pseudograph can be very large even it is 2-edge connected. (A 2-edge connected diameter 2 pseudograph has Betti deficiency at most 4.[9]) As to the graphs of diameter 4 or larger, some results are obtained in section 3.

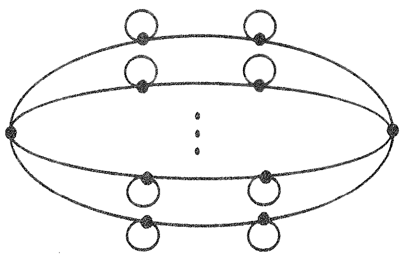

FIGURE 1.3

§ 2. The proof of Theorem 1.6, 1.7 and 1.8.

First, we need a lemma.

Lemma 2.1. *Let G be a connected graph with $\xi(G) \geq 2$ and let $A \subseteq E(G)$ be a minimal set such that $\nu(G, A) = \xi(G)$. Then*

(a) *$b(G-A) = c(G-A)$, and moreover, if G is a multigraph then every component of $G-A$ is non-trivial and if G is a simple graph then every component of $G-A$ contains at least three vertices;*
(b) *the end vertices of every edge in A belong to distinct components of $G-A$; and*
(c) *any two components of $G-A$ are joined by at most one edge of A.*

Proof. (a) Suppose not. Then there exists a component of $G - A$ with even Betti number; let it be F. Since $\xi(G) \geq 2$, we have $c(G - A) \geq 2$. Thus there is an edge $e \in A$ joining F and another component of $G - A$. Let $A' = A \backslash \{e\}$. Then $c(G - A') = c(G - A) - 1$, $b(G - A') = b(G - A)$, and $|A'| = |A| - 1$. This implies that

$$\begin{aligned}\nu(G, A') &= c(G - A') + b(G - A') - 1 - |A'| \\ &= c(G - A) - 1 + b(G - A) - 1 - (|A| - 1) \\ &= c(G - A) + b(G - A) - 1 - |A| \\ &= \nu(G, A) \\ &= \xi(G).\end{aligned}$$

Since A' is a proper subset of A, this contradicts to the minimality of A. Thus we have that every component of $G - A$ has odd Betti number. Moreover, if G is a multigraph then every component of $G - A$ is non-trivial and if G is a simple graph then every component of $G - A$ contains at least three vertices.

(b) Assume that f is an edge of A whose end vertices belong to the same component F_m. Let $A'' = A \backslash \{f\}$. Then the component in $G - A''$ which consists of F_m and f has even Betti number. Thus $c(G - A'') = c(G - A)$ and $b(G - A'') = b(G - A) - 1$. This implies that

$$\begin{aligned}\nu(G, A'') &= c(G - A'') + b(G - A'') - 1 - |A''| \\ &= c(G - A) + b(G - A) - 1 - |A| \\ &= \nu(G, A) \\ &= \xi(G).\end{aligned}$$

This contradicts to the fact that A is minimal and we have the proof of (b).

(c) Suppose not. Then there is a pair of components of $G - A$ such that they are joined by at least two edges of A. Let these two components be F_1 and F_2, and let e_1 and e_2 be two of the edges joining F_1 and F_2. Put $A''' = A \backslash \{e_1, e_2\}$. Then the component in $G - A'''$ which consists of F_1, F_2 and e_1, e_2 has odd Betti number. Thus $c(G - A''') = c(G - A) - 1$, $b(G - A''') = b(G - A) - 1$, and $|A'''| = |A| - 2$. Therefore we infer

$$\begin{aligned}\nu(G, A''') &= c(G - A''') + b(G - A''') - 1 - |A'''| \\ &= c(G - A) + b(G - A) - |A| \\ &= \nu(G, A) \\ &= \xi(G).\end{aligned}$$

Again, A''' is a proper subset of A which is not possible. Hence we have the proof of (c) and conclude the proof of Lemma 2.1. □

With the support of Lemma 2.1, we are able to construct a new graph based on the choice of A. Let G be a connected graph with $\xi(G) \geq 2$, and let A be a minimal set of $E(G)$ such that $\nu(G, A) = \xi(G)$. G_A is called a *testable graph* of G (with respect to A) if $V(G_A)$ is the set of components of $G - A$ and two vertices in G_A are adjacent whenever they are joined in G by one edge of A.

Accordingly, the following two lemmas are easy to prove.

Lemma 2.2. *If $\xi(G) \geq 2$ and G_A is a testable graph of G, then*

$$\xi(G) = \nu(G, A) = 2|V(G_A)| - |E(G_A)| - 1.$$

Proof. By the definition of G_A, $|V(G_A)| = c(G_A)$ and $|E(G_A)| = |A|$. Applying Lemma 2.1,

$$\begin{aligned}\xi(G) &= c(G-A) + b(G-A) - 1 - |A| \\ &= 2c(G-A) - 1 - |A| \\ &= 2|V(G_A)| - |E(G_A)| - 1. \quad \square\end{aligned}$$

Lemma 2.3. *If $\xi(G) \geq 2$ and G_A is a testable graph of G, then the minimum degree of G_A is not greater than 3, that is, $\delta(G_A) \leq 3$.*

Proof. Suppose not. Then $\deg_{G_A} F \geq 4$ for each vertex F in G_A. Thus $|E(G_A)| \geq 2|V(G_A)|$. By Lemma 2.2,

$$\xi(G) = 2|V(G_A)| - |E(G_A)| - 1 \leq 2|V(G_A)| - 2|V(G_A)| - 1 = -1.$$

This is a contradiction. So the proof is complete. $\square$

Now we are ready to prove Theorem 1.6.

The proof of Theorem 1.6.

Proof. Suppose that G is a multigraph of diameter 2 with $\xi(G) \geq 2$. By Lemma 2.1 and 2.3, there is a testable graph G_A with minimum degree at most 3. If G_A is a complete graph K_n, then by Lemma 2.2, $n = 2$ or 3. In each case, either (a) or (c) of Lemma 2.1 is violated. Thus G_A is not a complete graph. Hence there exists a pair of vertices F_1 and F_2 which are not adjacent in G_A. Let $\{a_i, b_i\} \subseteq V(F_i)$, $i = 1, 2$. ($V(F_i)$ denotes the vertex set of the component F_i in $G - A$.) Then the two vertices in each of the following pairs $\{a_1, a_2\}$, $\{b_1, b_2\}$, $\{a_1, b_2\}$ and $\{b_1, a_2\}$ must have a common neighbor outside $F_1 \cup F_2$, for otherwise the diameter of G is greater than 2. This implies that $\deg_{G_A} F_1 \geq 4$ and $\deg_{G_A} F_2 \geq 4$. Since G_A is simple, it has at least 5 vertices. But by Lemma 2.3, G_A contains a vertex H_1 of degree not larger than 3. This yields that there is a vertex H_2 not adjacent to H_1. By repeating the above argument for H_1 and H_2, we obtain $\deg_{G_A} H_1 \geq 4$, a contradiction. This concludes the proof. $\square$

The proof of Theorem 1.7.

Proof. Suppose not. Then $\xi(G) > 2$. By Lemma 2.1 and 2.3, there is a testable graph G_A with $\delta(G_A) \leq 3$. Let F_0 be a vertex of $V(G_A)$ which attains the minimum degree. We consider the following three cases.

Case 1. $\delta(G_A) = 1$. Let F_1 be the neighbor of F_0 in G_A. By the assumption that $\xi(G) > 2$, we have $c(G-A) \geq 3$ and thus there exists a vertex $F_2 \in V(G_A) \backslash \{F_0, F_1\}$. By Lemma 2.1, for each $F \in V(G_A)$, F contains at least two vertices in $G - A$. Therefore there exist $u_0 \in V(F_0)$ and $u_2 \in V(F_2)$ such that u_0 and u_2 are not adjacent to any vertex of $V(F_1)$. This implies that the distance of u_0 and u_2 is at least 4. But this is impossible for a graph with diameter 3. Hence we conclude that $\delta(G_A) = 1$ is not possible.

Case 2. $\delta(G_A) = 2$. Let F_1 and F_2 be the neighbors of F_0 in G_A. Thus, in G there are vertices $u_i \in V(F_i)$, $i = 0, 1, 2$, such that u_0u_1 and u_0u_2 are edges in G. Due to the fact that G_A is simple, these edges are the unique edges which join the component F_0 and components F_1 and F_2 respectively. Suppose that $|V(G_A)| = 3$, then F_1, F_2 must be adjacent in G_A. This yields that $\xi(G) = 2$ which is a contradiction to the assumption. Therefore $|V(G_A)| > 3$. Let the vertices of $V(G_A)$ be denoted by F_x, $x = 0, 1, 2, \cdots$. Now consider the components, F_i, $i > 2$. Since G is of diameter 3 and every component of $G - A$ is non-trivial, there exist at least two edge disjoint shortest paths $P_{i,0}$ and $P'_{i,0}$ of length 2 or 3 starting from the vertices of F_i to the vertices of F_0 in G. Let the set of starting edges of all shortest paths from the vertices of F_i to the vertices of F_0 be denoted by A_i. For convenience, any edge in A_i will be called a starting edge for F_i. Obviously, $A_i \subseteq A$ and $|A_i| \geq 2$. Consider $F_i, F_{i'} \in V(G_A)$ where $i \neq i'$ and $i, i' > 2$. If all the starting edges of F_i or $F_{i'}$ are incident with only the vertices of F_1 and F_2, then $A_i \cap A_{i'} = \emptyset$. Otherwise, there exists a shortest path $P_{i,0}$ of length 3 such that its starting edge e_i is incident with some vertex in F_k, $k > 2$. If $k \neq i'$, then $e_i \notin A_{i'}$. Assume that e_i is incident with a vertex of $F_{i'}$ and $e_i \in A_{i'}$. Then $P_{i,0}$ must be of length 2, which is a contradiction. Hence $|A_i \backslash A_{i'}| \geq 2$. This implies that $|A_i \backslash (\cup_{j \neq i, j > 2} A_j)| \geq 2$. For convenience, we shall assume that each set A_i, $i > 2$, contains exactly two starting edges which are not in any other $A_{i'}$, $i' > 2$ and $i \neq i'$. By a direct count, if there exists an edge $f \notin \{u_0u_1, u_0u_2\} \cup (\cup_{i>2} A_i)$ which joins two components of $G - A$, then

$$
\begin{aligned}
|A| &\geq 2 + 1 + 2(c(G-A) - 3) \\
&= 2c(G-A) - 3 \\
&= c(G-A) + b(G-A) - 3.
\end{aligned}
$$

This yields $\xi(G) \leq 2$, a contradiction and we conclude the proof of Case 2. In what follows, we shall claim that the edge mentioned above does exist.

First, if $F_1F_2 \in V(G_A)$, then we are done. Therefore assume that $F_1F_2 \notin V(G_A)$. Consider the distance of v_1, v_2 in G, $d(v_1, v_2)$, where $v_1 \in V(F_1)$, $v_2 \in V(F_2)$ and v_j is not incident with the vertices in F_0, $j = 1, 2$. In the case that $d(v_1, v_2) = 2$, there exists a vertex $v_k \in V(F_k)$, for some $k > 2$, such that v_kv_1 and v_kv_2 are in A. Since F_k is non-trivial, let $v'_k \in F_k \backslash \{v_k\}$. Thus, in order to keep that $d(v'_k, v_0) \leq 3$ for each $v_0 \in V(F_0)$, v'_k must be incident with an edge g in $A \backslash \{v_kv_1, v_kv_2\}$. Furthermore, g can not be a starting edge for $F_{k'}$, $k' > 2$ and $k' \neq k$. Hence g is the extra edge f we are looking for. On the other case, $d(v_1, v_2) = 3$. Thus there exists a shortest path of length 3 from v_1 to v_2, let it be $v_1 - a - b - v_2$. We consider the following three situations. First, $a \in V(F_1)$ or $b \in V(F_2)$. Then $d(a, v_2) = 2$ or $d(v_1, b) = 2$. By a similar argument as in the case $d(v_1, v_2) = 2$, we can find an edge f. Secondly, if a and b are in the same component F_k for some $k > 2$, then a must be adjacent to a vertex w in $F_{k'}$ for some $k' > 2$. For otherwise, there exists a vertex $u \in V(F_0)$ such that $d(a, u) \geq 4$. Therefore f can be found in $\{aw, av_1, bv_2\} \backslash A_k$. Finally, assume that a and b are in distinct components F_k, F_h, for some $k, h > 2$. If ab is not a starting edge of one of the two components, then let $f = ab$, we are done. On the other hand, let ab be a starting edge of a component F_h, $h > 2$. Without loss of generality, let $a \in V(F_h)$. Since F_h is non-trivial, let $a' \in V(F_h)\backslash\{a\}$. Clearly, a' is incident with an edge f' of A. Hence f can be found in $\{av_1, ab, f'\}\backslash A_h$, and thus we conclude the proof of Case 2.

Case 3. $\delta(G_A) = 3$. Let F_1, F_2, F_3 be the neighbors of F_0 in G_A. By Lemma 2.2, we have $2|V(G_A)| - |E(G_A)| - 1 > 2$ and $|E(G_A)| \geq \frac{3}{2}|V(G_A)|$. This implies $|V(G_A)| > 6$. As in Case 2, for each $x > 3$, let A_x be the set of starting edges of F_x such that $|A_x| = 2$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. A bit of reflection, if there exist two edges in A which are not incident with the vertex in F_0 and also not in any A_i, $i > 3$, then

$$\begin{aligned}|A| &\geq 2 + 2(c(G-A) - 4) + 3\\ &= 2c(G-A) - 3\\ &\geq c(G-A) + b(G-A) - 3.\end{aligned}$$

This implies $\xi(G) \leq 2$, a contradiction and we conclude that $\delta(G_A) = 3$ is not possible either. In what follows, we shall claim that the two edges mentioned above do exist.

To start with, if $H =< \{F_1, F_2, F_3\} >_{G_A}$ which is a subgraph of G_A induced by $\{F_1, F_2, F_3\}$ has size at least 2, then we are done. Therefore, there are two situations to consider. First, $|E(H)| = 1$. Without loss of generality, let $F_1F_2 \in E(H)$. Also, let $v_1 \in V(F_1)$ and $v_3 \in V(F_3)$ such that v_1 and v_3 are not adjacent to any vertex of F_0. Since $F_1F_3 \notin E(G_A)$, the shortest path from v_1 to v_3 must pass some vertices of F_i for some $i > 3$. Thus we can find an edge f as in the proof of Case 2. Therefore including the edge in H we have obtained the two edges. Secondly, $|E(H)| = 0$. Then $F_1F_2, F_2F_3, F_1F_3 \notin V(G_A)$. Since every component of $G - A$ is non-trivial, there exists a vertex $v_i \in V(F_i)$ for each $i = 1, 2, 3$ such that v_i is not adjacent to the vertices of $V(F_0)$. Moreover, since G is of diameter 3, for each j and k in $\{1, 2, 3\}$, $j \neq k$, there is a $v_j - v_k$ shortest path $P_{j,k}$ of length 2 or 3 such that $P_{j,k}$ contains no vertex of $V(F_0)$. Now let $\mathcal{F}_{j,k} = \{F_l | \, P_{j,k}$ contains a vertex in F_l, $l > 3\}$. Clearly, $|\mathcal{F}_{j,k}| = 1$ or 2. We consider the following three situations.

First, if there exist two distinct 2-subsets, $\{j, k\}$ and $\{j', k'\}$ of $\{1, 2, 3\}$ such that $\mathcal{F}_{j,k} \cap \mathcal{F}_{j',k'} = \emptyset$, then by the same argument as in Case 2, we can find an edge f for each 2-subset, thus we have the two edges.

Secondly, assume that $\mathcal{F}_{j,k} \cap \mathcal{F}_{j',k'} \neq \emptyset$ for any pairs $\{j, k\}, \{j', k'\} \subseteq \{1, 2, 3\}$, and there exists a set $\mathcal{F}_{j,k}$ with two elements. Without loss of generality, let $\mathcal{F}_{1,2} = \{F_x, F_y\}$, $x, y > 3$, and let $P_{1,2} = v_1 - a - b - v_2$ where $a \in V(F_x)$ and $b \in V(F_y)$. Now if $ab \in A_x \cup A_y$, say $ab \in A_x$. Then b must be adjacent to some vertex $v \in V(F_1) \cup V(F_3)\backslash\{v_1, v_3\}$ due to the reason that $ab \in A_x$ and then $d_G(b, u) \leq 2$ for some $u \in V(F_0)$. Again since each component in $G - A$ is non-trivial, we have that $c \in V(F_x)\backslash\{a\}$ and $d \in V(F_y)\backslash\{b\}$ which are incident with $e_1 \in A\backslash(\cup_{k\neq x}A_k)$ and $e_2 \in A\backslash(\cup_{k\neq y}A_k)$, respectively. Then the two edges can be found from the set $\{v_1a, ab, bv, bv_2, e_1, e_2\}$. (At most 4 of them are in $A_x \cup A_y$.) On the other hand, if $ab \notin A_x \cup A_y$. Let $f_1 = ab$. All we need is to find an edge f_2 which is not ab and not in any A_k, where $k > 2$. Since $\mathcal{F}_{1,2} \cap \mathcal{F}_{1,3} \neq \emptyset$, $F_x \in \mathcal{F}_{1,3}$ or $F_y \in \mathcal{F}_{1,3}$. If $\mathcal{F}_{1,3} = \{F_x, F_y\}$, then $P_{1,3}$ must be $v_1 - a - b - v_3$. Therefore, there exists an edge $g_1 \in A\backslash(\cup_{k \neq y} A_k)$ which is incident with a vertex of $V(F_y)\backslash\{b\}$. Then f_2 can be found in the set $\{v_2 b, v_3 b, g_1\}$. Otherwise, we can let $\mathcal{F}_{1,2} \cap \mathcal{F}_{1,3} = \{F_x\}$. If $\mathcal{F}_{1,3} = \{F_x\}$, then either $P_{1,3} = v_1 - a - v_3$ or $v_1 - a - c - v_3$ for some $c \in V(F_x)\backslash\{a\}$. Thus in any case, we can let c be a vertex of $V(F_x)\backslash\{a\}$. Since $d_G(c, u) \leq 3$ for each $u \in V(F_0)$, there exists an edge $g_2 \in A\backslash(\cup_{k \neq x} A_k)$ such that $g_2 \neq cv_3$ and g_2 is incident with c. Thus we can find $f_2 \in \{v_1 a, v_3 a, g_2\}$ or $\{v_1 a, v_3 c, g_2\}$ in respective cases. Finally, if $\mathcal{F}_{1,3} = \{F_x, F_z\}$ for some $z \notin \{0, 1, 2, 3, x, y\}$. Since the edge which joins F_x and F_z belogs to $A_x \cup A_z$, we can find an edge f_2 by a similar argument to that mentioned above.

Finally, we only have to check the situation when $\mathcal{F}_{1,2} = \mathcal{F}_{1,3} = \mathcal{F}_{2,3} = \{F_i\}$ for some $i > 3$; i.e., $P_{j,k}$ passes only the vertices of a fixed component F_i, $1 \leq j \neq k \leq 3$. Let h_j be the edge of A which joins F_j and F_i, $j \in \{1, 2, 3\}$. If h_1, h_2 and h_3 are incident with a common vertex a in F_i, then there is an edge $h_4 \in A\backslash(\cup_{k \neq i} A_k)$ which is incident with a vertex of $V(F_i)\backslash\{a\}$. Thus we have the two edges f_1, f_2 in $\{h_1, h_2, h_3, h_4\}$. Otherwise, h_1, h_2 and h_3 are not incident with a common vertex in F_i. This implies that there exists a vertex b in F_i such that b is incident with at most one edge of $\{h_1, h_2, h_3\}$. In order to keep $d_G(v_m, v_n) \leq 3$ for each pair $\{m, n\} \subseteq \{1, 2, 3\}$, h_j must be incident with v_j for each $j = 1, 2, 3$. Now since that $d_G(b, u) \leq 3$ for each $u \in V(F_0)$, we can find an edge $h_4 \in A\backslash(\cup_{k \neq i} A_k)$ such that $h_4 \notin \{h_1, h_2, h_3\}$ and h_4 is incident with b. Now the two edges can be found in $\{h_1, h_2, h_3, h_4\}$. This concludes the proof of Case 3 and the theorem. □

The proof of Theorem 1.8.

Proof. Since any graph in $\mathcal{S}$ or $\mathcal{T}$ has Betti deficiency 2, it suffices to show that if a diameter 3 simple graph is not upper embeddable then the graph is in either $\mathcal{S}$ or $\mathcal{T}$.

Let G be a diameter 3 simple graph which is not upper embeddable. By Lemma 2.1 and 2.3, there is a testable graph G_A with minimum degree not greater than 3. Now let $V(G_A) = \{F_0, F_1, \cdots, F_{|V(G_A)|-1}\}$, F_0 be a vertex of G_A with $\deg_{G_A} F_0 = \delta(G_A)$ and let $F_1, \cdots, F_{\delta(G_A)}$ be the neighbors of F_0. Consider F_i, $i > \delta(G_A)$. By Lemma 2.1, G_A is a simple graph and each component of $G - A$ contains at least three vertices. Since G is of diameter 3, there exist at least three edge disjoint paths of length 2 or 3 which start from the vertices of F_i to the vertices of F_0. (For otherwise, there are a vertex u in $V(F_0)$ and a vertex v in $V(F_i)$ such that $d_G(u, v) \geq 4$, a contradiction.) Similar to the proof of Theorem 1.7, let B_x be the set of starting edges of 3 disjoint paths of length 2 or 3 which start from the vertices of F_x to the vertices of F_0 such that $B_y \cap B_z \neq \emptyset$ for any $x, y, z > \delta(G_A)$ and $y \neq z$. Now if $\delta(G_A) = 3$ then by Corollary 1.5,

$$\lceil \frac{3c(G-A)}{2} \rceil = \lceil \frac{3|V(G_A)|}{2} \rceil \leq |E(G_A)| = |A|$$
$$< c(G-A) + b(G-A) - 2 \leq 2c(G-A) - 2.$$

This concludes $c(G-A) \geq 6$.

Assume that $c(G-A) > 6$. By the fact that $|B_x| = 3$ and $B_y \cap B_z = \emptyset$ for any $x, y, z > 3$, $y \neq z$, we obtain

$$|A| \geq 3(c(G-A)-4)+3 \geq 2c(G-A)-2 \geq c(G-A)+b(G-A)-2.$$

This contradicts to the assumption that $\xi(G) > 2$. Hence, $c(G-A) = 6$. Now if there exists a vertex $F_x \in V(G_A)$ such that $\deg_{G_A} F_x > 3$, then

$$\begin{aligned} |A| &\geq \lceil \frac{3|V(G_A)|+1}{2} \rceil = 10 = 2c(G-A)-2 \\ &\geq c(G-A)+b(G-A)-2. \quad (c(G-A)=6) \end{aligned}$$

This is impossible. Thus G_A must be a 3-regular graph. Again by the fact that $|B_x| = 3$, $B_y \cap B_z = \emptyset$ for any $x, y, z > 3$, $y \neq z$ and G_A is simple, it is easy to see that F_4, F_5 must be both adjacent to F_1, F_2, F_3. Therefore, G_A has to be the graph in Figure 2.1.

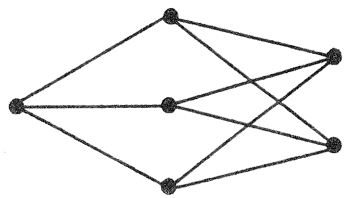

FIGURE 2.1

Assume that there exists a vertex $u \in V(F_i)$ such that u is not adjacent with any edge of A. Since G_A is vertex transitive, let $i = 0$. Thus for any $v \in V(F_j)$, $j = 4, 5$, v must be adjacent to one vertex of $V(F_k)$ for some $k \in \{1, 2, 3\}$ and this vertex is also adjacent to a vertex in $V(F_0)$. Due to the fact that G_A is simple, we can find $v_1 \in V(F_1)$ and $v_2 \in V(F_2)$ such that v_1 and v_2 are not incident with any edge of A. Clearly, $d_G(v_1, v_2) \geq 4$. This is not possible for a diameter 3 graph. Therefore, for any veretx $u \in V(G)$, u is incident with one edge of A. And then G must be as the graph in Figure 2.2. But this graph is of diameter 4. Hence we conclude that $\delta(G_A) \leq 2$.

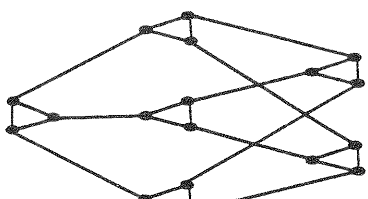

FIGURE 2.2

First if $\delta(G_A) = 1$, then it is clear that G_A is isomorphic to K_2. By Lemma 2.1, there exists an edge $e = v_1v_2$ such that $G - e = G_1 \cup G_2$ with $v_i \in V(G_i)$ and $\beta(G_i)$ odd for $i = 1, 2$. Furthermore, G is of diameter 3, so v_i must be adjacent to all vertices of $V(G_i)\backslash\{v_i\}$, for $i = 1, 2$. This implies that G is in $\mathcal{S}$.

Finally, we consider the situation that $\delta(G_A) = 2$. Again by the fact that $|B_x| = 3$ and $B_y \cap B_z = \emptyset$ for any $x, y, z > 3$, $y \neq z$, we infer

$$2c(G-A)-2 \geq c(G-A)+b(G-A)-2 > |A| \geq 2+3(c(G-A)-3) = 3c(G-A)-7.$$

This yields that $c(G - A) \leq 4$. Since G is of diameter 3 and $\delta(G_A) = 2$, we have that $c(G - A) = 3$ and G_A is isomorphic to K_3. Applying Lemma 2.1, A must be a 3-element edge subset in G such that $G - A = G_1 \cup G_2 \cup G_3$ and $\beta(G_i)$ odd for $i = 1, 2, 3$. Let $V_i = \{v \in V(G_i) | v$ is incident with an edge of $A\}$ for each $i = 1, 2, 3$. We claim that for each $v_i \in V_i$ and $w_i \in V(G_i)\backslash V_i$, v_i, w_i are adjacent in G.

Suppose not. Without loss of generality, let $v_1 \in V_1$ and $w_1 \in V(G_1)\backslash V_1$ such that v_1, w_1 are not adjacent. Clearly, v_1 must be adjacent to some vertex of V_2 or V_3. Without loss of generality, let v_1 be adjacent to a vertex of V_2. Then we can find a vertex $w_2 \in V(G_2)\backslash V_2$ such that $d(w_1, w_2) > 3$. This is a contradiction. Hence we finish the claim and conclude that G is in $\mathcal{T}$. □

§ 3. Concluding remarks

It would be interesting to know whether " Betti deficiency" is bounded from above for the graphs of diameter greater than 3. The graphs in Figure 3.1 show that $\xi(G)$ does not have an upper bound in general if G is a diameter 4 graph. Thus we only have to consider the graphs with edge connectivity 3. So far, not much results has been obtained in this direction except for the 3-edge connected graphs with diameter $k \geq 8$. The graph in Figure 3.2 shows that a 3-edge connected diameter 8 graph may have very large Betti deficiency. Therefore, it remains to consider $k \in \{3, 4, 5, 6, 7\}$ in the case where the graph is 3-edge connected.

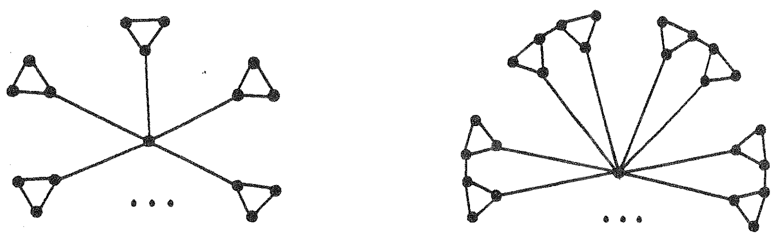

FIGURE 3.1. 1-edge connected and 2-edge connected

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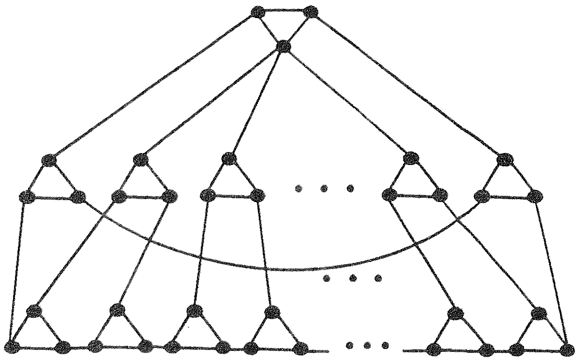

FIGURE 3.2. $k = 8$

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