On the Number of Indecomposable Block Designs

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Abstract. A t-(v,k,λ) design $\mathcal{D}$ is a system (multiset) of k-element subsets (called blocks) of a v-element set V such that every t-element subset of V occurs exactly λ times in the blocks of $\mathcal{D}$. A t-(v,k,λ) design $\mathcal{D}$ is called indecomposable (or elementary) if and only if there is no subsystem which is a t-(v,k,λ') design with $0<\lambda'<\lambda$. It is known that the number of indecomposable designs for given parameters t,v,k is finite. A block design is a t-(v,k,λ) design with $t=2$. The exact number of non-isomorphic, indecomposable block designs is only known for $k=3$ and $v\leq 7$. We computed the number of indecomposable designs for $v\leq 13$ and $\lambda\leq 6$. The algorithms used will be described.

1 Introduction

A t-(v,k,λ) *design* $\mathcal{D}$ is a system (multiset) of k-element subsets (called blocks) of a v-element set V such that every t-element subset of V occurs exactly λ times in the blocks of $\mathcal{D}$. A t-(v,k,λ) design $\mathcal{D}$ is called *indecomposable* (or elementary) if and only if there is no subsystem which is a t-(v,k,λ') design with $0<\lambda'<\lambda$. A survey about existence results was given by Archdeacon and Dinitz [1]. Two designs $\mathcal{D}$ and $\mathcal{D}'$ based on the same set V are called *isomorphic* if and only if there is a permutation of the elements of V which has the property that every block of $\mathcal{D}$ is bijectively mapped into a block of $\mathcal{D}'$; we write $\mathcal{D}'=\pi(\mathcal{D})$ with $\pi\in S_V$.
A *block design* is a t-(v,k,λ) design with $t=2$. The number of indecomposable t-(v,k,λ) designs for given parameters t,v,k is finite, see Street [17] or Engel [5]. One way to construct block designs with larger λ is to take the union of designs sharing a common set V. Let parameters t,v,k be fixed. The set of all non-isomorphic, indecomposable t-(v,k,λ) designs $\tilde{\mathcal{D}}^*[t,v,k]=\{\tilde{\mathcal{D}}_1^*,\tilde{\mathcal{D}}_2^*,\ldots,\tilde{\mathcal{D}}^*_{|\tilde{\mathcal{D}}^*[t,v,k]|}\}$ forms a finite generating system. Every t-(v,k,λ) design $\mathcal{D}$ can be built as follows:

$$\mathcal{D} = \biguplus_{i=1}^{|\tilde{\mathcal{D}}^*[t,v,k]|} \biguplus_{j=1}^{\alpha_i} \pi_{ij}(\tilde{\mathcal{D}}_i^*) \qquad \alpha_i\in\mathbb{N},\ \pi_{ij}\in S_V \quad ,$$

where α_i denotes how often the indecomposable design $\tilde{\mathcal{D}}_i^*\in\tilde{\mathcal{D}}^*[t,v,k]$ is to be used. The $\uplus$ sign means the union of multisets, i.e. if a block B occurs r times in

$\mathcal{D}_1$ and s times in $\mathcal{D}_2$, B occurs $s+t$ times in $\mathcal{D}_1 \uplus \mathcal{D}_2$. The exact number of non-isomorphic, indecomposable block designs is only known for $k=3$ and $v \leq 7$. In 1979 Burosch [2] showed that there exists exactly one indecomposable 2-$(6,3,\lambda)$ design. This is the only existing 2-$(6,3,2)$ design. Landgev [11] proved that the number of indecomposable 2-$(7,3,\lambda)$ designs is 2. One of them is the projective plane of order 2 (Fano plane). The second is obtained from all triples on 7 points by removing two disjoint Fano planes. A design is *simple* if it contains no repeated blocks. For a few small parameters the number of simple, indecomposable block designs is known. In Table 2 we show some results on simple designs and references.
Deciding whether a 2-$(v,k,2)$ design is decomposable can be done in polynomial time, see M. Colbourn [4]. C. Colbourn and M. Colbourn [3] proved that deciding whether a 2-$(v,3,\lambda)$ design (with $\lambda=3,4$) is decomposable is NP-complete.

2 Results

In 1993 Pietsch [16] developed a computer program called DESY which enumerates group divisible designs as the most general structures. We used DESY to construct block designs. The C++ program INDES is able to decide whether a design constructed in this way is decomposable or not. The computational results are presented in Table 1 (together with the best running time of INDES on a HP 735/125 workstation) and Table 2 (with reference INDES).
We introduce the following notations:

$ND(t,v,k,\lambda)$	is the number of non-isomorphic t-(v,k,λ) designs
$NE(t,v,k,\lambda)$	is the number of non-isomorphic, indecomposable t-(v,k,λ) designs
$NDC(t,v,k,\lambda)$	is the number of non-isomorphic, decomposable t-(v,k,λ) designs
$NSD(t,v,k,\lambda)$	is the number of simple, non-isomorphic t-(v,k,λ) designs
$NSE(t,v,k,\lambda)$	is the number of simple, non-isomorphic, indecomposable t-(v,k,λ) designs

$t-(v,k,\lambda)$	$ND(t,v,k,\lambda)$	$NE(t,v,k,\lambda)$	$NDC(t,v,k,\lambda)$	Time
2-(8,4,6)	2310	$1784^{S,D}$	526^{J}	3 min 26 s
2-(9,3,3)	22521	$13303^{S,D}$	9218^{J}	8 min 25 s
2-(9,4,6)	≥ 300953	$\geq 953^{S,D}$	$\geq 300000^{J}$	32 h
2-(10,4,4)	≥ 10733	$\geq 2849^{D}$	7884^{J}	2 h 39 min
2-(11,5,4)	4393	4298^{D}	95^{J}	1 h 45 min
2-(13,3,2)	≥ 311074	$\geq 61074^{D}$	$\geq 250000^{J}$	48 h
2-(13,4,2)	2461	2277^{D}	184^{J}	2 s

Table 1: Numbers of indecomposable and decomposable designs

In Tables 1 and 2 only nontrivial results are noted; e.g. it is trivial that $ND(t,v,k,\lambda) = NE(t,v,k,\lambda)$ if λ is the smallest number satisfying the well known necessary conditions. The numbers of designs listed in column $ND(t,v,k,\lambda)$ were taken from Pietsch [16]. It should be pointed out that the numbers of designs 2-(13, 4, 2) and 2-(8, 4, 6) in the listing of Mathon and Rosa [14] are incorrect. The corrected listing of design numbers will appear shortly in the CRC Handbook of Combinatorial Designs [13].
The capital letters after numbers in Table 1 denote the algorithms used. Here 'S' stands for 'Subset'-algorithm, 'D' for 'Decompose'-algorithm and 'J' stands for 'Join'-algorithm. The bold capital letter denotes the algorithm whose running time is given.

$t-(v,k,\lambda)$	$NSD(t,v,k,\lambda)$	$NSE(t,v,k,\lambda)$	Reference
2-(8,4,6)	164	128	[7],[8]
2-(8,4,9)	164	1	[7],[8]
2-(8,4,12)	4	0	[6]
2-(9,3,2)	13	11	[15],[12]
2-(9,3,3)	332	172	INDES,[10][1]
2-(9,3,4)	332	0	[10],INDES
2-(9,3,5)	13	0	[9]
2-(9,3,6)	1	0	[9]
2-(11,5,4)	3737	3679	INDES
2-(13,4,2)	1576	1453	INDES

Table 2: Numbers of simple, indecomposable designs

[1] Harnau's paper missed three designs; one of them is indecomposable.

3 The Algorithms used

The program DESY constructs one design from each isomorphism class. This representative is called the *canonical* design of that isomorphism class.
We used three different algorithms which can decide the question whether a design is indecomposable or decomposable.
The definition of indecomposability gives us the idea for the '*Subset*'-algorithm: For a given t-(v,k,λ) design $\mathcal{D}$ we have to find a permutation $\pi \in S_V$ and in the finite set of indecomposable, canonical t-(v,k,λ') designs ($\lambda' \leq \lfloor \frac{\lambda}{2} \rfloor$) a design $\tilde{\mathcal{D}}^*$, such that: $\pi(\tilde{\mathcal{D}}^*) \subset \mathcal{D}$. If we can not find such a design and such a permutation then $\mathcal{D}$ is indecomposable.

The '*Decompose*'-algorithm:
Let $\mathcal{D}$ be a t-(v,k,λ) design . We call the graph G with vertex set the blocks of $\mathcal{D}$ and edge set

$$E = \{(B_i B_j) : B_i, B_j \in \mathcal{D}\ , |B_i \cap B_j| \geq t \text{ and } i \neq j\}$$

the block-intersection graph of $\mathcal{D}$.
Looking at the block-intersection graph we can say:

Theorem 1

A t-(v,k,λ) design $\mathcal{D}$ based on a set V is decomposable if and only if there is a $\lambda' \in \mathbb{N}$ $(1 \leq \lambda' \leq \lfloor \frac{\lambda}{2} \rfloor)$ and a colouring (red,blue) of vertices (i.e. blocks) of the block-intersection graph such that for every pair $\{i,j\} \subseteq V$ there exist exactly λ' red coloured blocks which contain the pair $\{i,j\}$.

In the case $\lambda = 2$ such a colouring exists if and only if the block-intersection graph is bipartite. Then the 'Decompose'-algorithm can colour the graph in polynomial time. For $\lambda \geq 3$ we use a backtrack-algorithm to find a colouring. We orientated the description of the backtrack-algorithm to the notation which was used by Colbourn [4, p.75]. We have to decompose a t-(v,k,λ) design $\mathcal{D}$ which is given with any numbering of the blocks of $\mathcal{D}$. Let $1 \leq \lambda' \leq \lfloor \frac{\lambda}{2} \rfloor$. In the r-th step the algorithm has constructed a vector $x = (x_1, \ldots, x_r)$ (with integer $x_i \leq |\mathcal{D}|$ and $x_i \neq x_j$ for $i \neq j$). This vector denotes that the block x_k was coloured red in the k-th step. For testing of permissibility of the vector x we colour blue all yet uncoloured blocks of $\mathcal{D}$ which contain a pair $\{i,j\}$, which is contained in exactly λ' red coloured blocks. The vector x is permissible if there is no pair $\{i,j\}$ which is contained in more than λ' red coloured blocks or in more than $\lambda - \lambda'$ blue coloured blocks. If the vector x is not permissible we uncolour the block x_r and all blue coloured blocks. The set X_r contains all blocks which can occur in the r-th position of the vector x. If X_r is not empty we choose the first block for x_r and delete it from X_r. We make again the test of permissibility. If the set X_r is empty then it is necessary to backtrack to the previous component of the vector x and replace block x_{r-1}. If X_1 is empty the algorithm stops because there does not exist a permissible colouring with λ'. If the vector x is permissible then we search for a pair $\{i,j\}$ which is not contained in λ' red coloured blocks. If such a pair does not exist we have found a permissible colouring of the block-intersection graph. If such a pair $\{i,j\}$ exists then we form a new set X_{r+1} which has as elements all uncoloured blocks containing $\{i,j\}$. After choosing a block from X_{r+1} for component x_{r+1} and removing it from the set X_{r+1} we start again.

The '*Join*'-algorithm:
Our aim is to build all canonical, decomposable t-(v,k,λ) designs . A given canonical t-(v,k,λ) design $\mathcal{D}$ is indecomposable if and only if we can not find it in the set thus built.

Theorem 2

Every decomposable, canonical t-(v,k,λ) design $\mathcal{D}^*$ is isomorphic to a design $\mathcal{D}$, which can be built as the union of a canonical t-(v,k,λ_1) design $\mathcal{D}_1^*$ with an indecomposable t-(v,k,λ_2) design $\tilde{\mathcal{D}}_2$ with the property that $\lfloor \frac{\lambda}{2} \rfloor \geq \lambda_2$ and $\tilde{\mathcal{D}}_2$ is isomorphic to an indecomposable, canonical t-(v,k,λ_2) design $\tilde{\mathcal{D}}_2^*$.

$$\mathcal{D}^* \cong \mathcal{D} = \mathcal{D}_1^* \uplus \pi(\tilde{\mathcal{D}}_2^*), \text{ with } \lambda = \lambda_1 + \lambda_2 \text{ and } \pi \in S_V.$$

If we try all possible combinations of the union of a canonical design with a permuted, canonical, indecomposable design, we build every decomposable, canonical

t-(v,k,λ) design $\mathcal{D}^*$, up to isomorphism. For all these designs we construct the canonical design to make sure that we save only non-isomorphic designs.
A permutation $\pi \in S_V$ is called an automorphism of a design $\mathcal{D}$ if $\mathcal{D} = \pi(\mathcal{D})$. The set of all automorphisms of a design forms a group. This group is called the automorphism group of a design. We use automorphism groups for decreasing the running time of the 'Join'-algorithm.

Theorem 3
Let $\mathcal{D}_1$ be a t-(v,k,λ_1) design with automorphism group $Aut(\mathcal{D}_1)$ and let $\mathcal{D}_2$ be a t-(v,k,λ_2) design with automorphism group $Aut(\mathcal{D}_2)$. Then we have for all $\pi \in S_V$:

$$\mathcal{D}_1 \uplus \pi(\mathcal{D}_2) \cong \mathcal{D}_1 \uplus \pi_a \circ \pi \circ \pi_b(\mathcal{D}_2) \quad \forall \pi_a \in Aut(\mathcal{D}_1), \forall \pi_b \in Aut(\mathcal{D}_2)$$

During the 'Join'-algorithm we have a lot of permutations which generate the same design. For characterisation of these permutations we define an equivalence relation for elements $\pi_1, \pi_2 \in S_V$:

$$\pi_1 \sim \pi_2 \iff \pi_1 = \pi_a \circ \pi_2 \circ \pi_b \quad \pi_a \in Aut(\mathcal{D}_1), \pi_b \in Aut(\mathcal{D}_2).$$

So we only have to work with one representative from each equivalence class.

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