The intersection problem for a five vertex dragon.

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Abstract

The intersection problem for block designs asks for what values of k it is possible to find two designs (V, B_1) and (V, B_2), based on the same set V, with $|B_1 \cap B_2| = k$; that is, having precisely k common blocks.

We define a dragon design to be the decomposition of a complete graph into copies of graphs isomorphic to C_4 with a pendant edge. We solve the intersection problem for all such dragon designs.

1 Problem Overview

1.1 Background

Let G be a simple (undirected) graph which is some subgraph of another graph H. A *G-decomposition of H* is a block design (V, B) where V is the vertex set of H and B is an edge-disjoint decomposition of H into copies of G. In the case where H is K_n, the complete graph on n vertices, this is also called a *G-design of order n*.

The intersection problem for G-designs asks for what values of k it is possible to find two G-designs (V, B_1) and (V, B_2) (that is, two decompositions of a complete graph with vertex set V) with $|B_1 \cap B_2| = k$; that is, having precisely k common blocks.

Define $I_G(K_n)$ to be the set of achievable intersection sizes for a G-design of order n. It is easy to see that any two designs (V, B_1), (V, B_2) on the same vertex set with $|B_1| = |B_2| = b$ blocks can never have $b - 1$ blocks in common; that is, $b - 1 \notin I_G(K_n)$ for any G.

This problem has been considered for many combinatorial structures; a recent survey is [2]. In the specific case of small G-designs, where G is a graph with up to four vertices or up to four edges, a summary appears in [3].

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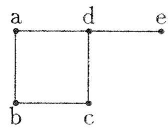
For the rest of this paper we look at decompositions of K_n into copies of a graph which we call a *dragon*. (The name *dragon* comes from a paper by Bermond, Huang, Rosa and Sotteau [1], where it is asserted that "this graph belongs to a class of graphs called dragons" and the reader is referred to a further paper by Huang [5]. However, the latter paper does not mention the word. For the sake of having something to call the graph, the name has been retained.) The dragon, denoted $\mathcal{D}$, is a cycle of length four plus a pendant edge:



It is important to note that the vertex at the free end of the pendant edge *is distinct from* the four vertices which form the cycle of length 4, so this is a graph with five vertices and five edges. We denote the above dragon by $[\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d} - \mathtt{e}]$ or equivalently $[\mathtt{c}, \mathtt{b}, \mathtt{a}, \mathtt{d} - \mathtt{e}]$, so that the edges used are $\{\mathtt{a}, \mathtt{b}\}$, $\{\mathtt{b}, \mathtt{c}\}$, $\{\mathtt{c}, \mathtt{d}\}$, $\{\mathtt{d}, \mathtt{a}\}$ and $\{\mathtt{d}, \mathtt{e}\}$.

If B denotes the set of all such blocks obtained from an edge-disjoint decomposition of K_n with vertex set V, then we denote the resulting design by $D = (V, B)$, calling it a *dragon design of order* n or equivalently a $\mathcal{D}$*-decomposition of* K_n.

The rest of this paper gives a solution to the intersection problem for this five-vertex dragon. We show that for all n for which dragon designs of order n exist, the set of intersection numbers is complete, i.e. $I_{\mathcal{D}}(K_n) = \{0, 1, \dots, b-2, b\}$, where b is the number of blocks in the design.

1.2 Methods

In order to find intersection numbers, two techniques are used, namely *permuting* and *trading*.

The former involves permuting the vertices of the original design. This is denoted by σD, where σ is the permutation and D a design.

The latter (also referred to in the literature as *mutually balanced sets*) involves replacing some of the blocks by a disjoint set of blocks which use precisely the same edges. We shall denote a trade by a table of the form

$$\begin{array}{c|c} [\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d} - \mathtt{e}] & [\alpha, \beta, \gamma, \delta - \epsilon] \\ [\mathtt{f}, \mathtt{g}, \mathtt{h}, \mathtt{i} - \mathtt{j}] & [\zeta, \eta, \theta, \iota - \kappa] \\ \vdots & \vdots \\ [\mathtt{k}, \mathtt{l}, \mathtt{m}, \mathtt{n} - \mathtt{o}] & [\lambda, \mu, \nu, \xi - \pi] \end{array}$$

where the blocks on the left are those in the original design which are to be replaced, and those on the right are their replacements, so the blocks on the right cover precisely the same edges as the blocks on the left (here, of course, $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \dots\} = \{\alpha, \beta, \gamma, \dots\}$). Moreover, if τ is some trade, then we use $|\tau|$ to refer to the number of blocks being traded, called the *volume* of the trade [4], and $\tau(D)$ to represent the trade applied to the design D.

Two other forms of notation are used to manipulate trades. Firstly, τ^i represents the trade τ cycled appropriately i times (i.e. with the vertices used in the trade permuted), where "appropriately" is defined in the context of the design to which the trade applies. Secondly, if τ and σ are two trades with edge-disjoint original block sets, we define $\tau \cup \sigma$ to be the union of the trades, with original block set equal to the union of those blocks from τ and σ and similarly for the final block set.

1.3 Necessary conditions

It is clear that for a dragon design of order n to exist, n must be 0 or 1 (mod 5). This is a consequence of the fact that the number of edges in the complete graph on n vertices, which is $\binom{n}{2} = n(n-1)/2$, must be a multiple of the number of edges in a dragon, which is 5.

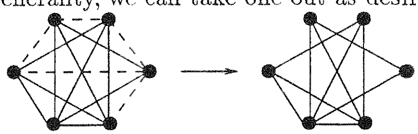
We see also that neither K_5 nor K_6 can be $\mathcal{D}$-decomposed:

Lemma 1 *There is no dragon design of order 5.*

Proof: K_5 has $\binom{5}{2} = 10$ edges, so a $\mathcal{D}$-decomposition would involve two dragons. But taking one out leaves no cycles of length more than 3, so another cannot be found. □

Lemma 2 *There is no dragon design of order 6.*

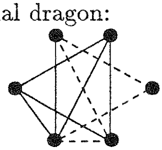
Proof: K_6 has $\binom{6}{2} = 15$ edges, so a $\mathcal{D}$-decomposition would involve three dragons. Without loss of generality, we can take one out as desired:



The vertices then (after removal of the dragon) have degrees 5, 4, 3, 3, 3 and 2.

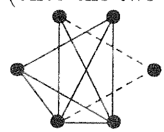
Now consider the vertex of degree 2. Either this vertex is part of a C_4 in one of the two remaining dragons in the decomposition, or else the pendant edges from each C_4 in the two remaining dragons are those edges leading to this vertex.

In the former case, there is only one possible way to form the cycle of length 4 containing the vertex, and removing those edges leaves no cycles of length greater than three for the final dragon:



Thus this does not lead to a dragon decomposition.

In the latter case, there do not exist two edge-disjoint cycles of length more than three in the remaining edges (once the two pendant edges are also removed):



Thus this also does not lead to a dragon decomposition. □

However, for all $n \geq 10$, these conditions are also sufficient; this was shown by Bermond, Huang, Rosa and Sotteau [1, 5]. We proceed from here by constructing dragon designs of order n for $n \equiv 0, 1 \pmod 5$, $n \geq 10$, and finding intersection numbers for all such designs.

2 Intersection Numbers

We begin by finding intersection sizes for pairs of dragon designs of orders 10, 11, 15 and 16, by looking at explicit decompositions of the appropriate complete graphs.

In order to proceed with the general construction we look also at $\mathcal{D}$-decompositions of $K_{5,5}$, the complete bipartite graph on 10 vertices.

We then construct dragon designs of order n for $n \equiv 0, 1, 5, 6 \pmod{10}, n \geq 20$; we do this by generalising from our explicit decompositions, deducing intersection sizes in the process.

2.1 Initial Cases

Table 1 gives appropriate parameters, vertex sets and starter blocks for dragon designs of orders 10, 11, 15 and 16. The set of possible intersection sizes for each of these designs is $I_{\mathcal{D}}(K_n) = \{0, 1, \ldots, b-2, b\}$, where b is the number of blocks in the design. The following tables give the permutations and trades used to find these values:

K_{10}:	Permutations in Table 2;	trades in Table 3.
K_{11}:	Permutations in Table 4;	trades in Table 5.
K_{15}:	Permutations in Table 6;	trades in Table 7.
K_{16}:	Permutations in Table 8;	trades in Table 9.

(The tables appear at the end of the paper.)

In the remainder of the paper, the notation D_n is used to refer to the dragon design of order n defined in Table 1.

2.2 $K_{5,5}$

Label the vertices of $K_{5,5}$ as `ij` to represent (i, j), where $i \in \{0, \ldots, 4\}$ and $j \in \{0, 1\}$.

Let D_5 be $[\texttt{01}, \texttt{00}, \texttt{11}, \texttt{30} - \texttt{21}]$ cycled modulo 5- (that is, first parameter modulo 5, second parameter fixed). Then D_5 is a $\mathcal{D}$-decomposition of $K_{5,5}$.

If E_5 is D_5 with bipartite sets swapped — that is, E_5 is generated by cycling $[00, 01, 10, 31 - 20]$ (modulo 5- as before) — then E_5 is also a $\mathcal{D}$-decomposition of $K_{5,5}$, and moreover $|E_5 \cap D_5| = 0$.

Hence we see that $I_{\mathcal{D}}(K_{5,5}) \supseteq \{0, 5\}$.

2.3 K_n, $n \equiv 0, 1, 5, 6 \pmod{10}$, $n \geq 20$

2.3.1 Designs

We construct $\mathcal{D}$-designs of order n for all remaining admissible n. The construction is of two forms, for $n \equiv 0, 1 \pmod{10}$ and $n \equiv 5, 6 \pmod{10}$.

- K_{10m}, K_{10m+1}:

Label the vertices of K_{10m} as $\mathbb{Z}_{2m} \times \mathbb{Z}_5$; adjoin a point ∞ for K_{10m+1}.

One design is then:

(1) For K_{10m}:
For each $i, 0 \leq i < m$, place a copy of a $\mathcal{D}$-design of order 10 onto the points $\{2i, 2i+1\} \times \mathbb{Z}_5$.
For K_{10m+1}:
For each $i, 0 \leq i < m$, place a copy of a $\mathcal{D}$-design of order 11 onto the points $(\{2i, 2i+1\} \times \mathbb{Z}_5) \cup \{\infty\}$.

(2) For each i, j with $0 \leq i < j < 2m$, when i is odd or $j > i+1$, place a copy of a $\mathcal{D}$-decomposition of $K_{5,5}$ onto the points $(\{i\} \times \mathbb{Z}_5) \cup (\{j\} \times \mathbb{Z}_5)$.

It is easy to see that this forms a valid decomposition of K_n by counting blocks and considering edges covered by each part of the decomposition.

- K_{10m+5}, K_{10m+6}:

Label the vertices of K_{10m+5} as $\mathbb{Z}_{2m+1} \times \mathbb{Z}_5$; include a point ∞ in the case of K_{10m+6}.

Define an equivalence relation $\rho \subseteq \mathbb{Z}_{2m+1} \times \mathbb{Z}_{2m+1}$ such that ρ is symmetric and for $i < j$, $(i, j) \in \rho$ if and only if $(i, j \in \{0, 1, 2\})$ or (i is odd and $j = i + 1$).

This partitions $\mathbb{Z}_{2m+1}$ into one class of size three and successive classes of size two (in increasing numerical order): $\{0, 1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2m-1, 2m\}$.

Label each class using the unique odd number in the class, i.e. let ρ_i denote the unique class containing i, where i is in $\mathbb{Z}_{2m+1}$ and is odd.

A suitable design is then:

(1) For K_{10m+5}:
  (a) For each $i, 1 \leq i < m$, place a copy of a $\mathcal{D}$-design of order 10 onto the points $\rho_{2i+1} \times \mathbb{Z}_5$.
  (b) Place a copy of a $\mathcal{D}$-design of order 15 onto the points $\rho_1 \times \mathbb{Z}_5$.

For K_{10m+6}:

(a) For each $i, 1 \leq i < m$, place a copy of a $\mathcal{D}$-design of order 11 onto the points $(\rho_{2i+1} \times \mathbb{Z}_5) \cup \{\infty\}$.

(b) Place a copy of a $\mathcal{D}$-design of order 16 onto the points $(\rho_1 \times \mathbb{Z}_5) \cup \{\infty\}$.

(2) For each i, j in different equivalence classes (i.e. $\rho_i \neq \rho_j$) place a copy of a $\mathcal{D}$-decomposition of $K_{5,5}$ onto the points $(\{i\} \times \mathbb{Z}_5) \cup (\{j\} \times \mathbb{Z}_5)$.

As before, it is easy to see that this is a valid decomposition of K_n by counting blocks and considering edges covered by each part of the decomposition.

2.3.2 Intersection Numbers

We now calculate the intersection numbers for these designs.

Define an operation $\oplus$ such that for two sets S_1 and S_2 of integers, $S_1 \oplus S_2$ is the set of all possible sums $s_1 + s_2$, where s_i is some element of S_i. Generalise this by defining $m * S$ to be $S_1 \oplus S_2 \oplus \ldots \oplus S_m$, where $S_i = S$ for $1 \leq i \leq m$, noting that associativity allows us to do so unambiguously. The multiplicative operation $*$ distributes over the additive operation $\oplus$ as usual.

We consider the solutions to each case $n \equiv 0, 1, 5, 6 \pmod{10}$ separately:

- K_{10m}

  We have $I(K_{10}) = \{0, 1, \ldots, 7, 9\}$ and $I(K_{5,5}) \supseteq \{0, 5\}$. Hence

$$\begin{aligned} I(K_{10m}) &\supseteq m * I(K_{10}) \oplus 4\binom{m}{2} * I(K_{5,5}) \\ &\supseteq m * \{0, 1, \ldots, 7, 9\} \oplus 4\binom{m}{2} * \{0, 5\} \\ &= \{0, 1, \ldots, 9m - 2, 9m\} \oplus 2m(m-1) * \{0, 5\} \\ &= \{0, 1, \ldots, 9m - 2, 9m\} \oplus \{0, 5, \ldots, 10m(m-1) - 5, 10m(m-1)\} \\ &= \{0, 1, \ldots, 10m^2 - m - 2, 10m^2 - m\}. \end{aligned}$$

  But $10m^2 - m$ is the number of blocks in a $\mathcal{D}$-decomposition of K_{10m}, so we know $10m^2 - m - 1 \notin I(K_{10m})$, and thus we have a complete set of intersection numbers.

- K_{10m+1}

  Here $I(K_{11}) = \{0, 1, \ldots, 9, 11\}$. Hence (in a similar fashion to the case of K_{10m} above)

$$\begin{aligned} I(K_{10m+1}) &\supseteq m * I(K_{11}) \oplus 4\binom{m}{2} * I(K_{5,5}) \\ &= \{0, 1, \ldots, 10m^2 + m - 2, 10m^2 + m\}. \end{aligned}$$

But $10m^2 + m$ is the number of blocks in a $\mathcal{D}$-decomposition of K_{10m+1}, so we know $10n^2 + n - 1 \notin I(K_{10n+1})$, and thus we again have a complete set of intersection numbers.

- K_{10m+5}
  Now $I(K_{15}) = \{0, 1, \ldots, 19, 21\}$. Hence

$$\begin{aligned} I(K_{10m+5}) &\supseteq I(K_{15}) \oplus (m-1) * I(K_{10}) \oplus (6(m-1) + 4\binom{m-1}{2}) * I(K_{5,5}) \\ &= \{0, 1, \ldots, 10m^2 + 9m, 10m^2 + 9m + 2\}. \end{aligned}$$

  But $10m^2 + 9m + 2$ is the number of blocks in a $\mathcal{D}$-decomposition of K_{10m+5}, so we know $10m^2 + 9m + 1 \notin I(K_{10m+5})$, and so again we have a complete set of intersection numbers.

- K_{10m+6}
  Here $I(K_{16}) = \{0, 1, \ldots, 22, 24\}$. Hence

$$\begin{aligned} I(K_{10m+6}) &\supseteq I(K_{16}) \oplus (m-1) * I(K_{11}) \oplus \left(6(m-1) + 4\binom{m-1}{2}\right) * I(K_{5,5}) \\ &= \{0, 1, \ldots, 10m^2 + 11m + 1, 10m^2 + 11m + 3\}. \end{aligned}$$

  But $10m^2 + 11m + 3$ is the number of blocks in a $\mathcal{D}$-decomposition of K_{10m+6}, so we know $10m^2 + 11m + 2 \notin I(K_{10m+6})$, and so we have a complete set of intersection numbers for this case also.

3 Conclusion

It is known that a dragon design of order n exists if and only if $n \equiv 0, 1 \pmod 5$ and $n \geq 10$ (refer to [1, 5]).

Here we have shown that the achievable intersection numbers for such a design are the complete set $\{0, 1, \ldots, b-2, b\}$, where $b = \binom{n}{2}/5$ is the number of blocks in a design of order n, so $n \equiv 0$ or $1 \pmod 5$, $n \geq 10$.

That is, we have proved:

Theorem *There exist two dragon designs $(V, B_1), (V, B_2)$ of order $|V| \equiv 0, 1 \ (mod\ 5)$, $|V| \geq 10$ with $|B_1 \cap B_2| = k$, for all $k \in \{0, 1, \ldots, b-2, b\}$ where b is the number of blocks in the design.*

References

[1] J.-C. Bermond, C. Huang, A. Rosa, and D. Sotteau, *Decomposition of complete graphs into isomorphic subgraphs with five vertices*, Ars Combinatoria **10** (1980), 211–254.

[2] E. J. Billington, *The intersection problem for combinatorial designs*, Congressus Numerantium **92** (1993), 33–54.

[3] E. J. Billington and D. L. Kreher, *The intersection problem for small G-designs*, Australasian J. Combinatorics **12** (1995), 239–258.

[4] A. Hedayat and S.-Y. R. Li, *The trade-off method in the construction of BIB designs with variable support sizes*, Annals of Statistics **7** (1979), 1277–1287.

[5] C. Huang, *Balanced graph designs on small graphs*, Utilitas Mathematica **10** (1976), 77–108.

Graph	No. of Blocks	Vertex set	Starter blocks
K_{10}	9	$\{\infty, 0, 1, \ldots, 8\}$	$[0, 2, 5, 4 - \infty]$ modulo 9
K_{11}	11	$\{0, 1, \ldots, 10\}$	$[2, 5, 1, 0 - 6]$ modulo 11
K_{15}†	21	$\{\infty, 0_1, 1_1, \ldots, 6_1, 0_2, 1_2, \ldots, 6_2\}$	$[0_1, 0_2, 2_2, 1_1 - \infty]$ $[3_2, 6_1, 1_2, 0_2 - \infty]$ $[2_1, 6_1, 4_1, 0_2 - 1_1]$ modulo 7 (subscripts fixed)
K_{16}‡	24	$\{\infty\} \cup (\mathbb{Z}_5 \times \mathbb{Z}_3)$ ij shorthand for (i, j)	$[30, 11, 01, 00 - \infty]$ $[40, 21, 11, 10 - \infty]$ $[00, 31, 21, 20 - \infty]$ $[10, 41, 31, 30 - \infty]$ $[41, 20, 40, 00 - 12]$ $[22, 10, 21, 00 - 32]$ $[32, 20, 31, 40 - \infty]$ $[42, 41, 02, 10 - 31]$ modulo -3 (i.e. fixed first position)

† An alternative decomposition can be found in [5].
‡ This decomposition is the one given in [1].

Table 1: Parameters for dragon decompositions of K_{10}, K_{11}, K_{15} and K_{16}.

σ	$\lvert D_{10} \cap \sigma D_{10} \rvert$
(05)	4
(02)	3
(01)	2
(012)	1
(0123)	0

Table 2: Intersection numbers for K_{10} from permutations.

$\|\tau\|$	τ		$\|D_{10} \cap \tau(D_{10})\|$
2	$[0,2,5,4-\infty]$	$[0,2,\infty,4-5]$	7
	$[7,0,3,2-\infty]$	$[3,0,7,2-5]$	
3	$[0,2,5,4-\infty]$	$[8,0,6,\infty-4]$	6
	$[2,4,7,6-\infty]$	$[0,2,5,4-6]$	
	$[4,6,0,8-\infty]$	$[7,6,2,4-8]$	
4	$\tau_2 \cup \tau_2^1$		5
Note that τ_j denotes the listed trade of size j.			

Table 3: Intersection numbers for K_{10} from trades.

σ	$\|D_{11} \cap \sigma D_{11}\|$
(06)	4
(04)	3
(02)	2
(01)(34)	1
(012)(34)	0

Table 4: Intersection numbers for K_{11} from permutations.

$\|\tau\|$	τ		$\|D_{11} \cap \tau(D_{11})\|$
2	$[2,5,1,0-6]$	$[0,2,5,1-7]$	9
	$[3,6,2,1-7]$	$[3,1,2,6-0]$	
3	$[2,5,1,0-6]$	$[5,0,8,6-7]$	8
	$[7,10,6,5-0]$	$[10,7,0,6-1]$	
	$[8,0,7,6-1]$	$[2,0,1,5-7]$	
4	$[2,5,1,0-6]$	$[4,3,5,8-2]$	7
	$[3,6,2,1-7]$	$[2,4,7,3-9]$	
	$[4,7,3,2-8]$	$[2,1,3,6-0]$	
	$[5,8,4,3-9]$	$[0,2,5,1-7]$	
5	$[2,5,1,0-6]$	$[5,8,7,2-0]$	6
	$[3,6,2,1-7]$	$[5,3,7,1-9]$	
	$[4,7,3,2-8]$	$[3,9,7,4-2]$	
	$[5,8,4,3-9]$	$[1,3,2,8-4]$	
	$[9,1,8,7-2]$	$[0,1,2,6-3]$	
6	$[2,5,1,0-6]$	$[4,7,5,8-2]$	5
	$[3,6,2,1-7]$	$[4,3,7,10-6]$	
	$[4,7,3,2-8]$	$[3,9,5,1-7]$	
	$[5,8,4,3-9]$	$[2,4,5,6-9]$	
	$[6,9,5,4-10]$	$[0,2,3,6-4]$	
	$[7,10,6,5-0]$	$[0,1,2,5-3]$	

Table 5: Intersection numbers for K_{11} from trades.

σ	$\lvert D_{15} \cap \sigma D_{15}\rvert$
$(0_1 2_1)$	11
$(0_1 6_2)$	10
$(0_1 1_1)$	9
$(0_1 2_1 1_2)$	8
$(0_1 1_1 2_1)$	7
$(0_2 \infty)$	6
$(0_1 1_1 0_2)$	5
$(0_1 1_1 0_2 1_2)$	4
$(0_1 \infty)$	3
$(0_1 0_2 \infty)$	2
$(0_1 1_1 2_1 3_1 4_1 5_1)$	1
$(0_1 1_1 0_2 \infty)$	0

Table 6: Intersection numbers for K_{15} from permutations.

$\lvert\tau\rvert$	τ		$\lvert D_{15} \cap \tau(D_{15})\rvert$
2	$[0_1, 0_2, 2_2, 1_1 - \infty]$ $[3_2, 6_1, 1_2, 0_2 - \infty]$	$[1_2, 6_1, 3_2, 0_2 - 2_2]$ $[0_1, 0_2, \infty, 1_1 - 2_2]$	19
3	$[0_1, 0_2, 2_2, 1_1 - \infty]$ $[1_1, 1_2, 3_2, 2_1 - \infty]$ $[3_2, 6_1, 1_2, 0_2 - \infty]$	$[1_2, 6_1, 3_2, 0_2 - 2_2]$ $[2_1, 3_2, 1_2, 1_1 - 2_2]$ $[1_1, 0_1, 0_2, \infty - 2_1]$	18
4	$\tau_2 \cup \tau_2^1$		17
5	$\tau_3 \cup \tau_2^2$		16
6	$\tau_4 \cup \tau_2^2$		15
7	$\tau_5 \cup \tau_2^3$		14
8	$\tau_4 \cup \tau_4^2$		13
9	$\tau_5 \cup \tau_4^3$		12
Note that τ_j denotes the listed trade of size j.			

Table 7: Intersection numbers for K_{15} from trades.

σ	$\lvert D_{16} \cap \sigma D_{16} \rvert$
(00 40)	14
(00 10)	13
(00 41)	12
(00 30)	11
(10 30)	10
(00 10 20)	9
(00 10 31)	8
(00 10 30)	7
(00 40)(10 30)	6
(00 10 20 30)	5
(00 10 20 30 40)	4
(00 10 20 01 02 12)	3
(00 10 20 30 40 01)	2
(00 10 30 01 02 12 42)	1
(00 10 20 30 40 01 11)	0

Table 8: Intersection numbers for K_{16} from permutations.

$\|\tau\|$	τ		$\|D_{16} \cap \tau(D_{16})\|$
2	$[12, 40, 30, 32 - \infty]$	$[12, 32, \infty, 40 - 30]$	22
	$[32, 20, 31, 40 - \infty]$	$[20, 31, 40, 32 - 30]$	
3	$[30, 11, 01, 00 - \infty]$	$[01, 31, 12, 02 - \infty]$	21
	$[31, 12, 02, 01 - \infty]$	$[10, 32, 02, 00 - \infty]$	
	$[32, 10, 00, 02 - \infty]$	$[00, 30, 11, 01 - \infty]$	
4	$[30, 11, 01, 00 - \infty]$	$[01, 31, 12, 02 - 00]$	20
	$[31, 12, 02, 01 - \infty]$	$[40, 21, 11, 10 - 00]$	
	$[32, 10, 00, 02 - \infty]$	$[10, 32, 02, \infty - 00]$	
	$[40, 21, 11, 10 - \infty]$	$[00, 30, 11, 01 - \infty]$	
5	$[30, 11, 01, 00 - \infty]$	$[41, 22, 12, 11 - 01]$	19
	$[31, 12, 02, 01 - \infty]$	$[31, 12, 02, 01 - 00]$	
	$[32, 10, 00, 02 - \infty]$	$[40, 21, 11, 10 - 32]$	
	$[40, 21, 11, 10 - \infty]$	$[00, 30, 11, \infty - 01]$	
	$[41, 22, 12, 11 - \infty]$	$[00, 10, \infty, 02 - 32]$	
6	$[30, 11, 01, 00 - \infty]$	$[32, 02, \infty, 10 - 41]$	18
	$[31, 12, 02, 01 - \infty]$	$[11, 12, 22, 41 - 31]$	
	$[32, 10, 00, 02 - \infty]$	$[01, 31, 12, 02 - 00]$	
	$[40, 21, 11, 10 - \infty]$	$[10, 40, 21, 11 - \infty]$	
	$[41, 22, 12, 11 - \infty]$	$[00, 01, 11, 30 - 31]$	
	$[10, 41, 31, 30 - \infty]$	$[00, 10, 30, \infty - 01]$	
7	$[30, 11, 01, 00 - \infty]$	$[32, 02, \infty, 10 - 11]$	17
	$[31, 12, 02, 01 - \infty]$	$[41, 20, \infty, 11 - 21]$	
	$[32, 10, 00, 02 - \infty]$	$[00, \infty, 01, 02 - 12]$	
	$[40, 21, 11, 10 - \infty]$	$[00, 41, 22, 12 - 11]$	
	$[41, 22, 12, 11 - \infty]$	$[00, 40, 21, 31 - 12]$	
	$[00, 31, 21, 20 - \infty]$	$[00, 30, 11, 01 - 31]$	
	$[41, 20, 40, 00 - 12]$	$[00, 10, 40, 20 - 21]$	
8	$\tau_2 \cup \tau_6$		16
9	$\tau_2 \cup \tau_7$		15
Note that τ_j denotes the listed trade of size j.			

Table 9: Intersection numbers for K_{16} from trades.

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