DECOMPOSITIONS OF COMPLETE MULTIPARTITE GRAPHS INTO SELFCOMPLEMENTARY FACTORS WITH FINITE DIAMETERS

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Abstract. For $r \geq 4$ we determine the smallest number of vertices, $g_r(d)$, of complete r-partite graphs that are decomposable into two isomorphic factors for a given finite diameter d. We also prove that for a given pair r, d such a graph exists for each order greater than $g_r(d)$.

1. Introductory notes and definitions

In this paper we study decompositions of finite complete multipartite graphs into two isomorphic factors with a prescribed diameter. A *factor* F of a graph $G = G(V, E)$ is a subgraph of G having the same vertex set V. A *decomposition* of a graph $G(V, E)$ into two factors $F_1(V, E_1)$ and $F_2(V, E_2)$ is a pair of factors such that $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E$. A decomposition of G is called *isomorphic* if $F_1 \cong F_2$. An isomorphism $\phi : F_1 \to F_2$ is then called a *complementing permutation* and the factors F_1 and F_2 the *selfcomplementary factors with respect to* G or simply the *selfcomplementary factors*. The *diameter* $\operatorname{diam} G$ of a connected graph G is the maximum of the set of distances $\operatorname{dist}_G(x, y)$ among all pairs of vertices of G. If G is disconnected, then $\operatorname{diam} G = \infty$. The *order* of a graph G is the number of vertices of G while the *size* of G is the number of its edges. For terms not defined here, see [1].

A. Kotzig and A. Rosa [7] and later P. Tomasta [9], D. Palumbíny [8], and P. Híc and D. Palumbíny [6] studied decompositions of complete graphs into isomorphic factors with a given diameter. E. Tomová [10] studied decompositions of complete bipartite graphs into two factors with given diameters and determined all possible pairs of diameters of such factors. T. Gangopadhyay [5] studied decompositions of complete r-partite graphs ($r \geq 3$) into two factors with given diameters and determined also all possible pairs of diameters of such factors.

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In this article we join both concepts. We study decompositions of complete r-partite graphs, for $r \geq 5$ into two isomorphic factors with a given diameter (for $r = 2, 3, 4$ see [3],[4]). We always assume that the number of vertices of an r-partite graph is at least $r + 1$, i.e. the graph is not a complete graph K_r.

T. Gangopadhyay [5] proved that a complete r-partite graph for $r \geq 3$ decomposable into two factors with the same finite diameter d exists if and only if $d = 2, 3, 4$ or 5. He also determined the smallest orders of such decomposable graphs.

A complete r-partite graph is *d-decomposable* if it is decomposable into two factors with the same finite diameter d. If we in addition require the factors to be mutually isomorphic, we say that the graph is *d-isodecomposable*. We also often say that a graph G is *isodecomposable* if it is d-isodecomposable for a finite diameter d which we do not determine specifically.

We show that there are d-isodecomposable complete r-partite graphs for each of the above mentioned diameters for any $r \geq 5$. In all cases we also present smallest decomposable graphs.

2. Preliminary theorems

We denote a complete r-partite graph with r partite sets having $m_1, m_2, \ldots, m_r$ vertices, respectively, as $K_{m_1,m_2,\ldots,m_r}$. Or, especially if there are more parts having the same cardinality, we denote the complete r-partite graph having k_i parts of cardinality n_i for $i = 1, 2, \ldots, s$ by $K_{n_1^{k_1} n_2^{k_2} \ldots n_s^{k_s}}$. In this case we always suppose that $k_1 + k_2 + \cdots + k_s = r$ and $n_i \neq n_j$ for $i \neq j$.

Let $f_r(d)$ denote the smallest number of vertices of a complete r-partite d-decomposable graph. If such a number does not exist, then we define $f_r(d) = \infty$.

It is obvious that any d-isodecomposable complete r-partite graph $K_{m_1,m_2,\ldots,m_r}$ must have an even number of edges and hence the number of parts having odd cardinalities must be 0 or 1 (mod 4). A graph with this property as well as the corresponding r-tuple $m_1, m_2, \ldots, m_r$ is called *admissible*.

We can similarly introduce $g_r(d)$ as the smallest number of vertices of a complete d-isodecomposable r-partite graph. We also define $g'_r(d)$ as the smallest integer with the property that for any $n \geq g'_r(d)$ there is a complete r-partite d-isodecomposable graph with n vertices. Finally, we define $h_r(d)$ as the smallest integer such that any admissible complete r-partite graph with at least $h_r(d)$ vertices is d-isodecomposable. If such numbers do not exist, we again put $g_r(d) = \infty$, $g'_r(d) = \infty$ or $h_r(d) = \infty$, respectively. It is obvious that

$$f_r(d) \leq g_r(d) \leq g'_r(d) \leq h_r(d).$$

The first and last inequality can be in some cases sharp. For instance, Gangopadhyay [5] proved that $f_r(2) = r + 1$, but we show that $g_r(2) = r + 1$ only if $r \equiv 1$ or $2 (\bmod 4)$ while $g_r(2) = r + 2$ for $r \equiv 0 (\bmod 4)$ and $g_r(2) = r + 3$ for $r \equiv 3 (\bmod 4)$. The last inequality can be sharp as well: for $r \equiv 0 (\bmod 4)$ it holds that $g_r(5) = g'_r(5) = r + 5$, but $h_r(5) = \infty$. It is an immediate consequence of the following result.

First we need some definitions. Let $N = \{1, 2, \ldots, n\}$. Two sequences $B = b_1, b_2, \ldots, b_n$ and $C = c_1, c_2, \ldots, c_n$ are *isomorphic* if there exists a one-to-one mapping $\psi : N \to N$ such that $b_i = c_{\psi(i)}$. The *degree sequence* of a graph G with a vertex set $v_1, v_2, \ldots, v_n$ is the sequence $A = a_1, a_2, \ldots, a_n$ where $a_i = \deg v_i$. The sequence is *isodecomposable* if there exist isomorphic sequences $B = b_1, b_2, \ldots, b_n$ and $C = c_1, c_2, \ldots, c_n$ such that $a_i = b_i + c_i$ for each $i \in N$. Obviously, a graph G is isodecomposable only if the degree sequence of G is isodecomposable. Moreover, G is isodecomposable into two factors with a finite diameter only if the degree sequence of G is isodecomposable into two sequences with all positive entries.

Theorem 1. *Let $l, m, r, s; r \neq s$ be odd numbers. Then the graph $K_{r^l s^m}$ is not d-isodecomposable for any d.*

Proof. The degree sequence of $K_{r^l s^m}$ is $p, p, \ldots, p, q, q, \ldots, q$ where both numbers $p = (l-1)r + ms$ and $q = lr + (m-1)s$ are odd and both appear in the sequence an odd number of times, namely p appears $lr = t$ times and q appears $ms = n - t$ times. Suppose, to the contrary, that $K_{r^l s^m}$ is isodecomposable. We may assume without loss of generality that $p < q$. Let $A = a_1, a_2, \ldots, a_n$ and $B = b_1, b_2, \ldots, b_n$ be isomorphic sequences such that $a_i + b_i = p$ for $i = 1, 2, \ldots, t$ and $a_i + b_i = q$ for $i = t+1, t+2, \ldots, n$. Let $\alpha(i)$ ($\beta(i)$) for $i = 0, 1, \ldots, p$ be the number of terms of $a_1, a_2, \ldots, a_t$ ($b_1, b_2, \ldots, b_t$) which are equal to i and $\alpha'(j)$ ($\beta'(j)$) for $i = 0, 1, \ldots, q$ be the number of terms of $a_{t+1}, a_{t+2}, \ldots, a_n$ ($b_{t+1}, b_{t+2}, \ldots, b_n$) which are equal to j. Obviously, $\alpha(i) = \beta(p - i)$ and $\alpha'(i) = \beta'(q - i)$.

Because t is odd, there must be i such that $\alpha(i) > \beta(i)$. Let i_0 be the smallest number i such that $\alpha(i) > \beta(i)$. Denote $k = \alpha(i) - \beta(i)$. As the sequences A and B are isomorphic, i_0 must appear in $b_{t+1}, b_{t+2}, \ldots, b_n$ k-times more than in $a_{t+1}, a_{t+2}, \ldots, a_n$, i.e., $\beta'(i_0) - \alpha'(i_0) = k$. Then $\alpha'(q - i_0) - \beta'(q - i_0) = k$, i.e., $q - i_0$ appears more often in $a_{t+1}, a_{t+2}, \ldots, a_n$ than in $b_{t+1}, b_{t+2}, \ldots, b_n$. Hence $q - i_0$ must appear in $b_1, b_2, \ldots, b_t$ k more times than in $a_1, a_2, \ldots, a_t$, which yields $\beta(q - i_0) - \alpha(q - i_0) = k$. This is equivalent to $\alpha(i_0 + p - q) - \beta(i_0 + p - q) = k$. Because $k > 0$, we have $\alpha(i_0 + p - q) > \beta(i_0 + p - q)$. From the minimality of i_0 it follows that $i_0 + p - q \geq i_0$, which contradicts our assumption that $p < q$ and therefore $K_{r^l s^m}$ is not isodecomposable. □

Corollary 2. *$h_r(d) = \infty$ for every $r \equiv 0 \pmod 4$ and any d.*

Proof. Given any $r \equiv 0 \pmod 4$ and any order n, we can construct an infinite class of graphs $K_{2n+1,(4n+1)^{r-1}}$ with order greater than n. Since $r - 1$ is an odd number, the graph $K_{2n+1,(4n+1)^{r-1}}$ is not d-isodecomposable by Theorem 1 and hence $h_r(d) = \infty$ for any d. □

On the other hand, we prove later that $g_r(d) = g'_r(d)$ for each $r \geq 5$ and each possible finite d. This equality was proven to be true also for $r = 2, 3, 4$ and all finite diameters in [3],[4].

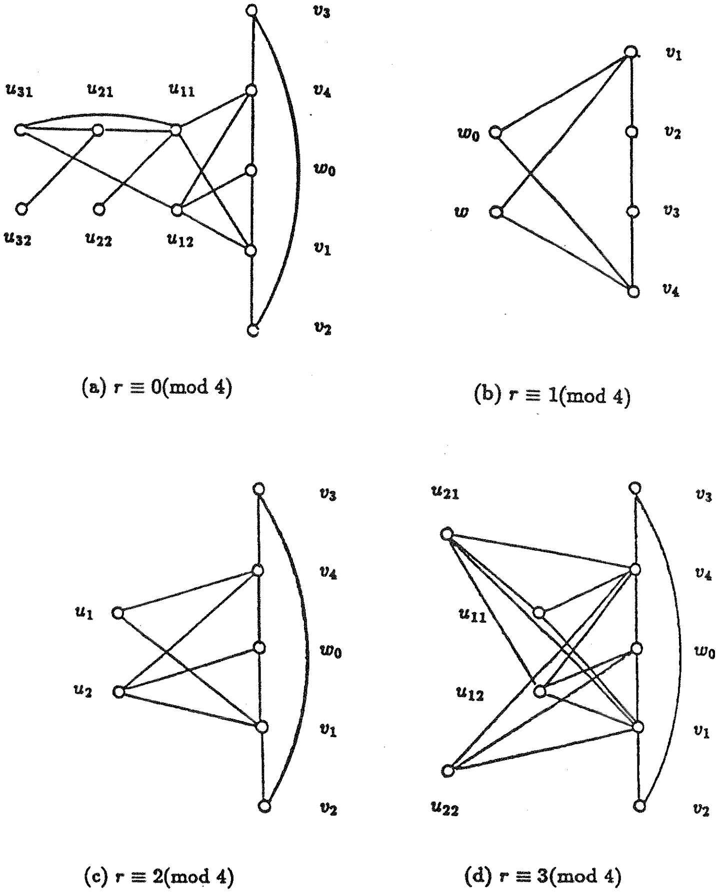


Figure 1

3. Constructions

In this section we construct d-isodecomposable complete r-partite graphs of the smallest orders for every $r \geq 5$ and every possible finite diameter d.

Construction 3. *(a)* Case $r \equiv 0 (\mathrm{mod}\, 4)$. For $r = 8$ we take the graph shown in Figure 1.a. To get a selfcomplementary factor of $K_{2,2,2,1,\ldots,1}$ with parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}\}, U_2 = \{u_{21}, u_{22}\}, U_3 = \{u_{31}, u_{32}\}, V_i = \{v_i\}, i = 1, 2, 3, 4$, we add all edges $u_{21}x$ and $u_{31}x$ for $x \in \{w_0, v_1, v_2, v_3, v_4\}$ whenever the edge $u_{11}x$ exists and all edges $u_{22}x$ and $u_{32}x$ whenever the edge $u_{12}x$ exists. The complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{21}u_{22}), (u_{31}u_{32}),$ $(v_1v_3v_4v_2)$. For any $r = 4k+8, k \geq 1$, we add parts $V_5, V_6, \ldots, V_{4k+4}$, where $V_j = \{v_j\}$. Then for every quadruple $v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}$ we add the edges of the path $P_4 = \langle v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}\rangle$, i.e., $v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3}, v_{4i+3}v_{4i+4}$, and join the end-vertices v_{4i+1} and v_{4i+4} of P_4 to all "preceding" vertices, i.e., to the vertices $u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}, w_0, v_1, v_2, \ldots, v_{4i}$. The new cycles of ϕ are then $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$.

(b) Case $r \equiv 1 (\mathrm{mod}\, 4)$. For $r = 5$ we take the selfcomplementary factor shown in Figure 1.b. The parts of $K_{2,1,1,1,1}$ are $W = \{w, w_0\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$, the complementing permutation ϕ is determined by the cycles $(w_0), (w), (v_1v_3v_4v_2)$. For any $r = 4k+5, k \geq 1$, we add again vertices $v_5, v_6, \ldots, v_{4k+4}$ (or, more precisely, parts $V_j = \{v_j\}$) and for every quadruple $v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}$ we add the edges of the path $P_4 = \langle v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}\rangle$, i.e., $v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3}, v_{4i+3}v_{4i+4}$, and join the end-vertices v_{4i+1} and v_{4i+4} of P_4 to all "preceding" vertices, i.e., to the vertices $w_0, w, v_1, v_2, \ldots, v_{4i}$. The new cycles of ϕ are now again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$.

(c) Case $r \equiv 2 (\mathrm{mod}\, 4)$. For $r = 6$ we take the selfcomplementary factor shown in Figure 1.c. The complementing permutation ϕ is determined by the cycles $(w_0), (u_1u_2), (v_1v_3v_4v_2)$. For any $r = 4k+6, k \geq 1$, we add again vertices (i.e., parts,) $v_5, v_6, \ldots, v_{4k+4}$ and all the edges as in the case (b). The new cycles of ϕ are again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$.

(d) Case $r \equiv 3 (\mathrm{mod}\, 4)$. For $r = 7$ we take the selfcomplementary factor shown in Figure 1.d. The complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{21}u_{22}), (v_1v_3v_4v_2)$. For any $r = 4k+7, k \geq 1$, we again add the vertices, edges and permutation cycles as in the previous cases. □

We continue with smallest 3-isodecomposable graphs for each $r \geq 5$. The construction is in all cases very similar to the previous one. We again take first the r-partite factors for $r = 5, 6, 7, 8$ and extend them by adding paths P_4, but we join to the "preceding" vertices the inner vertices of P_4 rather than the end-vertices.

Construction 4. *(a)* Case $r \equiv 0 (\mathrm{mod}\, 4)$. For $r = 8$ we take the graph shown in Figure 2.a. To get a selfcomplementary factor of $K_{2,2,2,1,\ldots,1}$ with parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}\}, U_2 = \{u_{21}, u_{22}\}, U_3 = \{u_{31}, u_{32}\}, V_i = \{v_i\}, i = 1, 2, 3, 4$, we add all edges $u_{21}x$ and $u_{31}x$ for $x \in \{w_0, v_1, v_2, v_3, v_4\}$ whenever the edge $u_{11}x$ exists and all edges $u_{22}x$ and $u_{32}x$ whenever the edge $u_{12}x$ exists. The complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{21}u_{22}),$ $(u_{31}u_{32}), (v_1v_3v_4v_2)$. For any $r = 4k+8, k \geq 1$, we add parts $V_5 = \{v_5\}, V_6 = \{v_6\}, \ldots, V_{4k+4} = \{v_{4k+4}\}$. Then for every quadruple $v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}$ we add the edges of the path $P_4 = \langle v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}\rangle$, namely $v_{4i+1}v_{4i+2}$, $v_{4i+2}v_{4i+3}, v_{4i+3}v_{4i+4}$, and join the inner vertices v_{4i+2} and v_{4i+3} of P_4 to all "preceding" vertices, i.e., to the vertices $u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, u_{32}, w_0, v_1, v_2, \ldots,$ v_{4i}. The new cycles of ϕ are then $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$. The vertices at distance 3 apart are always u_{31} and u_{32}.

(b) Case $r \equiv 1 (\mathrm{mod}\, 4)$. For $r = 5$ we take the selfcomplementary factor shown in Figure 2.b. The parts of $K_{2,1,1,1,1}$ are $W = \{w, w_0\}, V_1 = \{v_1\}, V_2 =$

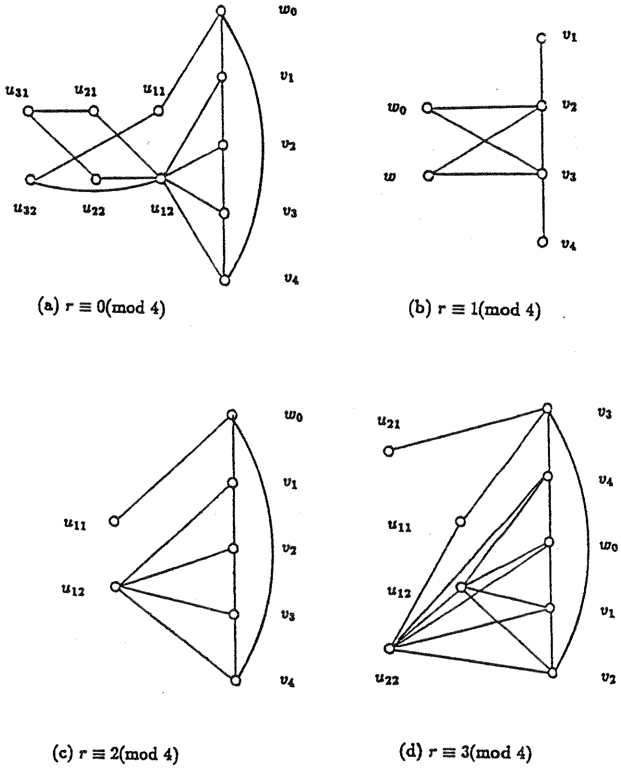


Figure 2

$\{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$, the complementing permutation ϕ is determined by the cycles $(w_0), (w), (v_1 v_3 v_4 v_2)$. For any $r = 4k + 5, k \geq 1$, we add again vertices $v_5, v_6, \ldots, v_{4k+4}$ (i.e., parts $V_j = \{v_j\}$) and for every quadruple $v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4}$ we add again the edges of $P_4 = \langle v_{4i+1}, v_{4i+2}, v_{4i+3}, v_{4i+4} \rangle$, i.e., $v_{4i+1}v_{4i+2}$, $v_{4i+2}v_{4i+3}$, $v_{4i+3}v_{4i+4}$, and join the inner vertices v_{4i+2} and v_{4i+3} of P_4 to all "preceding" vertices, i.e., to the vertices $w_0, w, v_1, v_2, \ldots, v_{4i}$. The new cycles of ϕ are now again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$. The vertices having mutual distance 3 are v_{k+1} and v_{4k+4}.

(c) Case $r \equiv 2 (\mathrm{mod}\, 4)$. For $r = 6$ we take the selfcomplementary factor

shown in Figure 2.c. The complementing permutation ϕ is determined by the cycles $(w_0), (u_1u_2), (v_1v_3v_4v_2)$. For any $r = 4k+6, k \geq 1$, we add again vertices (parts) $v_5, v_6, \ldots, v_{4k+4}$ and all the edges as in the case (b). The new cycles of ϕ are again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$. The vertices u_{11} and u_{12} are always at distance 3.

(d) Case $r \equiv 3(\bmod 4)$. For $r = 7$ we take the selfcomplementary factor shown in Figure 2.d. The complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{21}u_{22}), (v_1v_3v_4v_2)$. For any $r = 4k+7, k \geq 1$, we again add the vertices, edges and permutation cycles as in the previous cases. □

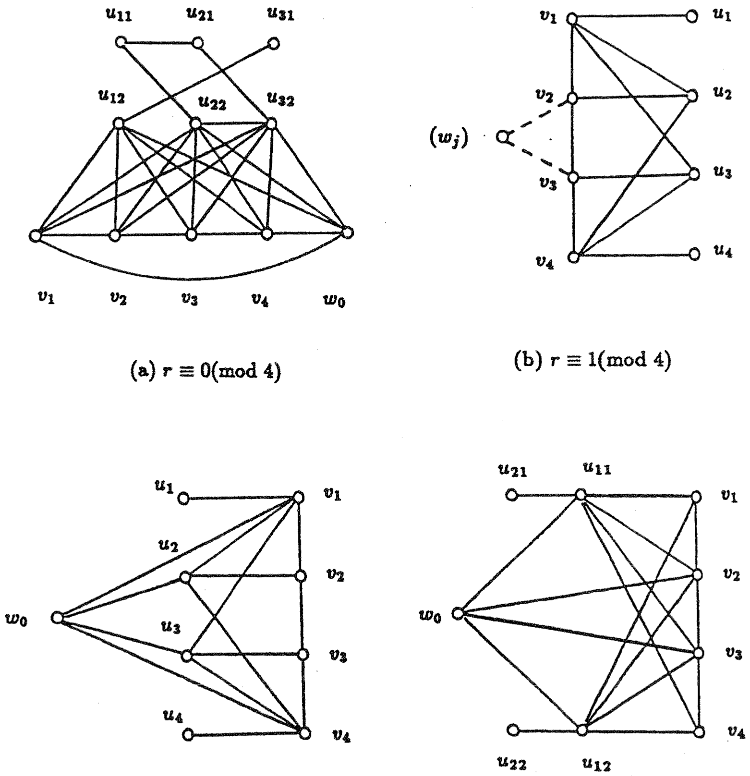
In constructions of factors with diameters 4 and 5 we use a different approach. To increase the number of parts, we "blow up" the path P_4 induced by vertices belonging to different trivial parts.

First we construct smallest selfcomplementary factors with diameter 4 of the complete r-partite graphs for each $r \geq 5$.

Construction 5. *(a)* Case $r \equiv 0(\bmod 4)$. We start with decomposition of the 8-partite graph $K_{2,2,2,1,\ldots,1}$ with the parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}\}, U_2 = \{u_{21}, u_{22}\}, U_3 = \{u_{31}, u_{32}\}, V_i = \{v_i\}, i = 1, 2, 3, 4$. The selfcomplementary factor is shown in Figure 3.a. The complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{21}u_{22}), (u_{31}u_{32}), (v_1v_3v_4v_2)$. The vertices having distance 4 are u_{11} and u_{31}. For any $r = 4k+8, k \geq 1$, we add parts $V_j = \{v_j\}, i = 5, 6, \ldots, 4k+4$. Now we "blow up" the path $P_4(0) = \langle v_1, v_2, v_3, v_4 \rangle$. We add the edges of the paths $P_4(i)$, namely $v_{4i+1}v_{4i+2}, v_{4i+2}v_{4i+3}, v_{4i+3}v_{4i+4}$ for $i = 1, 2, \ldots, k$, all edges $v_{4i+1}v_{4l+2}, v_{4i+2}v_{4l+3}, v_{4i+3}v_{4l+4}$ and all edges $v_{4i+2}v_{4l+2}$ and $v_{4i+3}v_{4l+3}$ for all pairs $i, l \in \{0, 1, \ldots, k\}, i \neq l$. We also add the edges $v_{4i+r}x$ for all $i = 1, 2, \ldots, k$ and $r = 1, 2, 3, 4$ whenever the edge v_rx exists. Here x is any vertex of $W \cup U_1 \cup U_2 \cup U_3$. In other words, we take the path $P_4(0) = \langle v_1, v_2, v_3, v_4 \rangle$, put the vertices $v_{4i+r}, i = 0, 1, \ldots, k; r = 1, 2, 3, 4$ "into" the vertex v_r and substitute the original edge v_rv_{r+1} for all possible edges $v_{4i+r}v_{4l+r+1}$. The vertices v_{4i+2} and $v_{4i+3}, i = 0, 1, \ldots, k$ induce complete graphs K_{k+1}, while the vertices v_{4i+1} and $v_{4i+1}, i = 0, 1, \ldots, k$ remain mutually non-adjacent. Finally, every vertex v_{4i+r} has the same neighbours in $W \cup U_1 \cup U_2 \cup U_3$ as the vertex v_r. One can check that u_{11} and u_{31} are at distance 4. The new cycles of ϕ are now $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$.

(b) Case $r \equiv 1(\bmod 4)$. We first decompose the graph $K_{4,1,1,1,1}$ with parts $U = \{u_1, u_2, u_3, u_4\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ into factors isomorphic to the one shown in Figure 3.b. (The indicated vertex w_j appears later in the construction of graphs of greater orders.) The complementing permutation is determined by the cycles $(u_1u_3u_4u_2)$ and $(v_1v_3v_4v_2)$. For any $r = 4k+5, k \geq 1$, we add the parts $V_5 = \{v_5\}, V_6 = \{v_6\}, \ldots, V_{4k+4} = \{v_{4k+4}\}$ and "blow up" the path $P_4(0) = \langle v_1, v_2, v_3, v_4 \rangle$ exactly as in part (a). The new cycles of ϕ are again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$. The vertices having mutual distance 4 are u_1 and u_4.

(c) Case $r \equiv 2(\bmod 4)$. We start with the graph $K_{4,1,1,1,1}$ with parts $W = \{w_0\}, U = \{u_1, u_2, u_3, u_4\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ and decom-



(a) $r \equiv 0 (\mathrm{mod}\ 4)$ (b) $r \equiv 1 (\mathrm{mod}\ 4)$

(c) $r \equiv 2 (\mathrm{mod}\ 4)$ (d) $r \equiv 3 (\mathrm{mod}\ 4)$

Figure 3

pose it into factors isomorphic to the factor shown in Figure 3.c. The complementing permutation ϕ is determined by the cycles $(w_0), (u_1u_3u_4u_2)$ and $(v_1v_3v_4v_2)$. For any $r = 4k+6, k \geq 1$, we again "blow up" the path $P_4(0) = \langle v_1, v_2, v_3, v_4 \rangle$ exactly as in part (a), adding the parts $V_5 = \{v_5\}, V_6 = \{v_6\}, \ldots, V_{4k+4} = \{v_{4k+4}\}$ and the corresponding edges. The new cycles of ϕ are again $(v_{4i+1}v_{4i+3}v_{4i+4}v_{4i+2})$ for $i = 1, 2, \ldots, k$. The vertices at distance 4 apart are u_1 and u_4.

(d) Case $r \equiv 3(\mathrm{mod}\,4)$. For $r = 7$ we decompose the graph $K_{2,2,1,1,1,1,1}$ with parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}\}, U_2 = \{u_{21}, u_{22}\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ into factors isomorphic to the graph in Figure 3.d. The comple-

menting permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{22}), (u_{12}u_{21})$ and $(v_1v_3v_4v_2)$. We increase the number of parts for any $r = 4k+3$ as in the previous cases. The vertices having mutual distance 4 are u_{21} and u_{22}. □

Finally, we construct factors of smallest 5-isodecomposable complete r-partite graphs for each $r \geq 5$.

Construction 6. In this construction we present only the factors of smallest 5-isodecomposable complete r-partite graphs with $r = 5, 6, 7, 8$ and 9 parts. The factors of smallest graphs for any $r \geq 10$ can be obtained exactly as in Construction 5—by "blowing up" the path $P_4(0) = \langle v_1, v_2, v_3, v_4 \rangle$.

(a) Case $r \equiv 0 \pmod 4$. The 8-partite graph $K_{4,2,2,1,\ldots,1}$ with the parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}, u_{13}, u_{14}\}, U_2 = \{u_{21}, u_{22}\}, U_3 = \{u_{31}, u_{32}\}, V_i = \{v_i\}, i = 1, 2, 3, 4$, is 5-isodecomposable into the selfcomplementary factors shown in Figure 4.a. The complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{13}u_{14}), (u_{21}u_{22}), (u_{31}u_{32}), (v_1v_3v_4v_2)$. The vertices having mutual distance 4 are u_{11} and u_{14}.

(b) Case $r \equiv 1 \pmod 4$. The 5-partite graph $K_{4,2,2,2,1}$ with the parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}, u_{13}, u_{14}\}, U_2 = \{u_{21}, u_{22}\}, U_3 = \{u_{31}, u_{32}\}, U_4 = \{u_{41}, u_{42}\}$ is 5-isodecomposable into the selfcomplementary factors isomorphic to the subgraph of the graph shown in Figure 3.b induced by the above mentioned parts. The complementing permutation ϕ is determined by the cycles $(w_0), (u_{11}u_{12}), (u_{13}u_{14}), (u_{21}u_{22}), (u_{31}u_{32}), (u_{41}u_{42})$. The vertices at mutual distance 5 are u_{11} and u_{14}.

To obtain the selfcomplementary factor of the complete 9-partite graph $K_{4,2,2,2,1,\ldots,1}$, we have to add to the graph in Figure 4.b the parts $V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ and edges v_iu_{jl} for each $i = 1, 2, 3, 4$ whenever the edge w_0u_{jl} exists. The permutation ϕ contains now one more cycle, $(v_1v_3v_4v_2)$.

(c) Case $r \equiv 2 \pmod 4$. The factor of the 6-partite graph $K_{4,2,1,1,1,1}$ with the parts $W = \{w, w_0\}, U = \{u_1, u_2, u_3, u_4\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ is shown in Figure 4.c. The cycles of ϕ are $(w), (w_0), (u_1u_3u_4u_2), (v_1v_3v_4v_2)$ and the vertices at distance 5 are u_1 and u_4.

(d) Case $r \equiv 3 \pmod 4$. The 7-partite graph $K_{4,2,1,\ldots,1}$ with the parts $W = \{w_0\}, U_1 = \{u_{11}, u_{12}, u_{13}, u_{14}\}, U_2 = \{u_{21}, u_{22}\}, V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{v_3\}, V_4 = \{v_4\}$ is 5-isodecomposable into the factors isomorphic to that in Figure 4.d. The cycles of ϕ are $(w_0), (u_{11}u_{12}), (u_{13}u_{14}), (u_{21}u_{22}), (v_1v_3v_4v_2)$ and the vertices at distance 5 are u_{11} and u_{14}. □

4. Small graphs: negative results

We start with Gangopadhyay's result [5] on decomposability into factors (not necessarily isomorphic) with the same finite diameter.

Theorem 7. (Gangopadhyay) *Let a complete r-partite graph $K_{m_1, m_2, \ldots, m_r}$ with more than 4 parts be d-decomposable for a finite diameter d. Then $d = 2, 3, 4$ or 5*

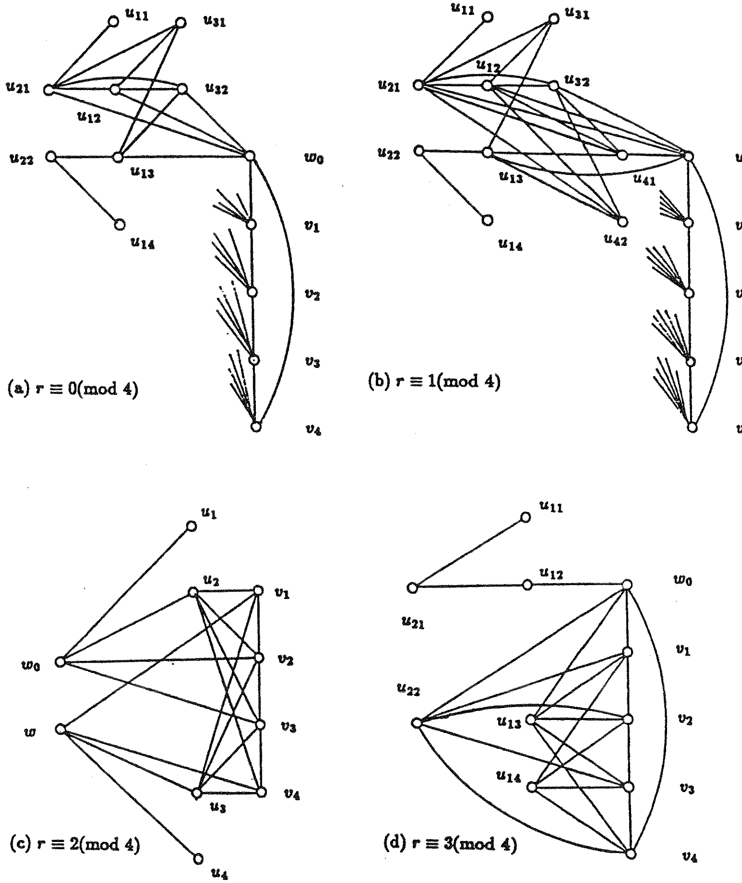


Figure 4

and

(a) $m_1 + m_2 + \cdots + m_r \geq r + 1$ *if* $d = 2$;
(b) $m_1 + m_2 + \cdots + m_r \geq r + 1$ *if* $d = 3$;
(c) $m_1 + m_2 + \cdots + m_r \geq r + 2$ *if* $d = 4$;
(d) $m_1 + m_2 + \cdots + m_r \geq r + 4$ *if* $d = 5$.

Our main goal is to prove the following.

Theorem 8. *Let* $r \geq 5$. *Then*
$g_r(2) = g_r(3) = g_r(4) = r + 3,\ g_r(5) = r + 5$ *if* $r \equiv 0 (\mathrm{mod}\, 4)$,

$g_r(2) = g_r(3) = r+1,\ g_r(4) = r+3,\ g_r(5) = r+6$ *if* $r \equiv 1 (\bmod 4)$,
$g_r(2) = g_r(3) = r+1,\ g_r(4) = r+2,\ g_r(5) = r+4$ *if* $r \equiv 2 (\bmod 4)$, *and*
$g_r(2) = g_r(3) = g_r(4) = r+2,\ g_r(5) = r+4$ *if* $r \equiv 3 (\bmod 4)$.

From Theorem 7 it follows that for every $r > 4, r \equiv 2(\bmod 4)$ and each $d = 2,3,4,5$, every $r \equiv 1(\bmod 4)$ and $d = 2,3$, and every $r \equiv 3(\bmod 4)$ and $d = 4,5$ all d-decomposable complete r-partite graphs are also d-isodecomposable. In the other cases we need to show that there are no isodecomposable graphs of smaller orders. To do this, we need the following lemma, proved in [4].

The *neighbourhood* of a vertex x in a graph G, denoted $N_G(x)$, is a set of all vertices adjacent to x in G. If A is a set of vertices of G, then $N_G(A)$ is the union of neighbourhoods of all vertices of A.

Lemma 9. *Let* $K_{m_1,m_2,\dots,m_r}, r \geq 3$, *be 5-isodecomposable into factors* F_1 *and* F_2. *Let* A_i *be the set of all vertices of excentricity 5 in* F_i *and* $\phi : F_1 \to F_2$ *be a complementing permutation. Then* $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 \subset V_j$ *and* $N_{F_1}(A_1) = N_{F_2}(A_2)$, *or equivalently* $\phi(N_{F_1}(A_1)) = N_{F_1}(A_1)$. *Moreover, if* V_k *is the partite set containing* $N_{F_1}(A_1)$, *then* $\phi(V_k) = V_k$.

The following two theorems are immediate corollaries of the Lemma.

Theorem 10. *Let* $K_{m_1,m_2,\dots,m_r}$ *be 5-isodecomposable and let* $r \geq 3; m_1 \geq m_\geq \dots \geq m_r$. *Then* $m_1 \geq 4$ *and* $m_2 \geq 2$.

For $r \equiv 0(\bmod 4)$ an additional condition holds. Although the proof of the theorem can be found in [4], we include it here for the sake of completeness and because we refer to it later.

Theorem 11. *Let* $r \equiv 0(\bmod 4)$ *and* $K_{m_1,m_2,\dots,m_r}$ *be 5-isodecomposable. Then at least 3 of the cardinalities* $m_1, m_2, \dots, m_r$ *must be even.*

Proof. We need to show only that one of the numbers $m_1, m_2, \dots, m_r$ must be even, because if just one or two of them are even, then $K_{m_1,m_2,\dots,m_r}$ has an odd number of edges. Let $N_{F_1}(A_1) \subset V_r$. If $|V_r|$ is even, we are done. From Lemma 9 it follows that $\phi(V_r) = V_r$ and hence $\phi(V_1 \cup V_2 \cup \dots \cup V_{r-1}) = V_1 \cup V_2 \cup \dots \cup V_{r-1}$. Then obviously $K_{m_1,m_2,\dots,m_r}$ is isodecomposable only if the graph $K_{m_1,m_2,\dots,m_{r-1}}$ is isodecomposable. This is possible only if the number of odd parts is either 0 or $1(\bmod 4)$ which implies that at least two of the numbers $m_1, m_2, \dots, m_{r-1}$ must be even. But $|V_r|$ was odd and hence the actual number of even cardinalities among $m_1, m_2, \dots, m_{r-1}$ must be at least 3. □

Now we can exclude the small non-isodecomposable graphs.

Lemma 12. *Let* $r > 4, r \equiv 0$ *or* $3(\bmod 4)$. *Then there is no d-isodecomposable graph* $K_{m_1,m_2,\dots,m_r}$ *with* $r+1$ *vertices.*

Proof. Obviously, there is only one complete r-partite graph with $r+1$ vertices, namely $K_{2,1,1,\dots,1}$, which is not admissible for $r \equiv 0$ or $3(\bmod 4)$. □

Lemma 13. *Let $r > 4, r \equiv 0 (\mathrm{mod}\, 4)$. Then there is no d-isodecomposable graph $K_{m_1,m_2,\ldots,m_r}$ with $r+2$ vertices for any d.*

Proof. There are only two possible complete r-partite graphs with $r+2$ vertices, namely $K_{2,2,1,1,\ldots,1}$ and $K_{3,1,1,\ldots,1}$. The former is not admissible, while the latter is not isodecomposable by Theorem 1. □

Lemma 14. *Let $r > 4, r \equiv 0 (\mathrm{mod}\, 4)$. Then there is no 5-isodecomposable graph $K_{m_1,m_2,\ldots,m_r}$ with less than $r+5$ vertices.*

Proof. By Theorem 8, every 5-isodecomposable r-partite complete graph has one part of cardinality at least 4 and another of cardinality at least 2 and by Theorem 12 it contains at least three even parts. Obviously, $K_{4,2,2,1,\ldots,1}$ with $r+5$ vertices is the smallest graph satisfying these conditions. □

Lemma 15. *Let $r > 4, r \equiv 1 (\mathrm{mod}\, 4)$. Then there is no 4-isodecomposable graph $K_{m_1,m_2,\ldots,m_r}$ with $r+2$ vertices.*

Proof. There are two graphs $K_{m_1,m_2,\ldots,m_r}$ with $r+2$ vertices. $K_{2,2,1,1,\ldots,1}$, which is not admissible, and $K_{3,1,1,\ldots,1}$. Let us suppose that there is $r \equiv 1 (\mathrm{mod}\, 4)$ such that the r-partite graph $K_{3,1,1,\ldots,1}$ is $(2,4)$-isodecomposable into factors F_1 and F_2. Let $U = \{u_1, u_2, u_3\}$ be one part and $V_i = \{v_i\}, i = 1, 2, \ldots, r-1$ the other parts, and let $V = V_1 \cup V_2 \cup \cdots \cup V_{r-1}$.

We first assume that there is a pair of vertices u_i, u_j, say u_1, u_2, such that $\mathrm{dist}_{F_1}(u_1, u_2) = 4$. Obviously, $N_{F_1}(u_1) \cup N_{F_1}(u_2) \subset V$ and $N_{F_1}(u_1) \cap N_{F_1}(u_2) = \emptyset$. Furthermore, there is no edge between $N_{F_1}(u_1)$ and $N_{F_1}(u_2)$. Let $M = V \backslash N_{F_1}(u_1) \backslash N_{F_1}(u_2)$. Then in F_2 all vertices of $N_{F_1}(u_1)$ are adjacent to u_2, all vertices of $N_{F_1}(u_2)$ are adjacent to u_1, and each vertex of $N_{F_1}(u_1)$ is adjacent to all vertices of $N_{F_1}(u_2)$. If the vertices u_1 and u_2 have no common neighbour in F_2, i.e., if $M = \emptyset$, the diameter of the graph $\langle V \cup u_1 \cup u_2 \rangle_{F_2} = F_2 - u_3$ is 3 and the only vertices having excentricity 3 in this graph are u_1 and u_2. Since u_3 is not adjacent to either of them, we can see that $\mathrm{ex}_{F_2}\, v_3 \leq 3$, which yields $\mathrm{diam}\, F_2 \leq 3$. If $M \neq \emptyset$, then the diameter of the graph $\langle V \cup u_1 \cup u_2 \rangle_{F_2} = F_2 - u_3$ is 2 and therefore again $\mathrm{diam}\, F_2 \leq 3$. Thus if $\mathrm{dist}_{F_i}(x, y) = 4$, at least one of the vertices x, y belongs to V.

Now we show that if $\mathrm{dist}_{F_i}(x, y) = 4$ and $x = v_i \in V$ then $y \notin V$. Suppose it is not the case and there are vertices of V, say v_1, v_2, such that $\mathrm{dist}_{F_1}(v_1, v_2) = 4$. Denote F_i' the subgraph of F_i induced by the vertices of V. Then clearly $\mathrm{diam}\, F_1' \geq 4$. It is well known that if a factor of a complete graph K_n has diameter greater than 3, then its complement (with respect to K_n) has diameter at most 2. Because $\langle V \rangle = K_{r-1}$, the diameter of F_2' is at most 2. Then all vertices with excentricity 4 in F_2 belong to U, which is impossible by the preceding paragraph.

Thus we have only one possibility left, namely that there are vertices u_i and v_j, say u_1, v_1, such that $\mathrm{dist}_{F_1}(u_1, v_1) = 4$. Then $\langle V \cup u_1 \rangle \cong K_r$ and the graph $\langle V \cup u_1 \rangle_{F_2}$ has diameter at most 2, because $\mathrm{diam} \langle V \cup u_1 \rangle_{F_1} \geq 4$. Hence the only vertices which could have excentricity 4 in F_2 are u_2 and u_3. Then $\mathrm{dist}_{F_2}(u_2, u_3) = 4$, which is a contradiction completing the proof. □

Lemma 16. *Let $r > 4, r \equiv 1 (\bmod 4)$. Then there is no 5-isodecomposable graph $K_{m_1,m_2,\ldots,m_r}$ with less than $r+6$ vertices.*

Proof. By Theorem 8 every 5-isodecomposable graph $K_{m_1,m_2,\ldots,m_r}$ contains $K' = K_{4,2,1,1,\ldots,1}$. This graph has $r+4$ vertices and is not admissible for $r \equiv 1 (\bmod 4)$.

There are only 3 graphs or order $r+5$, containing K'. The first one, $K_{4,2,2,1,\ldots,1}$, is not admissible. Let us investigate then the graph $K_{5,2,1,1,\ldots,1}$ and denote the part with 5 vertices by V_1, and the part with 2 vertices by V_2. It follows from Lemma 9 that the vertices which have excentricity 4 in either factor belong to V_2 and the complementing permutation, ϕ, takes V_2 onto itself. Hence, similarly as in the proof of Theorem 11 the r-partite graph $K_{5,2,1,1,\ldots,1}$ is isodecomposable only if the $(r-1)$-partite graph $K_{5,1,1,\ldots,1}$ is isodecomposable. But $K_{5,1,1,\ldots,1}$ has $r-2$ trivial parts, which is an odd number, and therefore is not d-isodecomposable for any d by Theorem 1.

The last case, $K_{4,3,1,1,\ldots,1}$, is similar. By the same arguments as above, ϕ takes the part with 3 vertices onto itself and $K_{4,3,1,1,\ldots,1}$ is isodecomposable only if the $(r-1)$-partite graph $K_{4,1,1,\ldots,1}$ is isodecomposable, too. But for $r \equiv 1 (\bmod 4)$ the graph $K_{4,1,1,\ldots,1}$ with $r-2$ parts of cardinality 1 is not admissible, and therefore $K_{4,3,1,1,\ldots,1}$ is not 5-isodecomposable. □

The proof of our main result is now straightforward.

Proof of Theorem 8. Apply Lemmas 12–16 and Constructions 3–6. □

Now we are ready to prove that once there exists an d-isodecomposable complete r-partite graph, $r \geq 5$, of order p_0 then such a graph exists for each order greater than p_0.

Theorem 17. *Let $5 \leq r < \infty$. Then $g_r(d) = g'_r(d)$ for any finite d.*

Proof. $g_r(d) = \infty$ for $d = 1$ or $5 < d < \infty$, hence the result is immediate.

To prove the assertion for any d, $2 \leq d \leq 5$ we need to show that for a given d and any $p \geq g_r(d)$ there is a complete r-partite d-isodecomposable graph with p vertices. Let $p = g_r(d) + q$. For $d = 4$ and $r \equiv 1 (\bmod 4)$ we take the factor constructed in part (b) of Construction 5, add q vertices $w_1, w_2, \ldots, w_q$ into part U and join each of them in the factor F_1 to all vertices $v_{4i+2}, v_{4i+3}, i = 1, 2, \ldots, k$. Then $\phi(w_j) = w_j$ for each $j = 1, 2, \ldots, q$ and obviously $F_1 \cong F_2$. In all other cases one can see that $\phi(w_0) = w_0$. Therefore we can always add q vertices $w_1, w_2, \ldots, w_q$ into part W and join in F_1 each of them to all neighbours of w_0. Then again $\phi(w_j) = w_j$ for each $j = 1, 2, \ldots, q$ and $F_1 \cong F_2$. □

Let us remark that the equality holds also for $r = 2, 3, 4$ (see [3],[4]).

5. Concluding remarks

We have found the smallest orders of complete r-partite graphs that can be decomposed into selfcomplementary factors with a given finite diameter. However, the spectrum of all such graphs is yet to be determined.

P. Das [2] introduced the following classes of graphs. A complete graph without one edge, $\tilde{K}_n = K_n - e$, is called an *almost complete graph.* A graph G with n vertices is *almost selfcomplementary* if the graph $\tilde{K}_n$ can be decomposed into two factors that are both isomorphic to G. We can similarly introduce a class of *almost complete multipartite graphs*, i.e. the complete multipartite graphs with one missing edge. Then we can study decompositions of such graphs into two isomorphic factors, called *almost selfcomplementary factors.*

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