Ternary Designs With The Maximum Or Near Maximum Number of Replication Numbers

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Abstract

Ternary designs with replication numbers all or all but one of the values in the range $\Lambda(V-1)/(K-1)$ to $\Lambda(V-1)/(K-2)$ are studied. It is known that for K=3 and replication numbers all values in the range $\Lambda(V-1)/(K-1)$ to $\Lambda(V-1)/(K-2)$, designs exist if and only if $\Lambda=2$. Here we show the uniqueness of such designs. When K=3 and all values in the range $\Lambda(V-1)/(K-1)$ to $\Lambda(V-1)/(K-2)$ but one are used, designs are constructed for $\Lambda=2$ and all V except for one case. Nonexistence is shown for the missing case. Ternary designs with even block size and replication numbers all or all but one of the values in the range $\Lambda(V-1)/(K-1)$ to $\Lambda(V-1)/(K-2)$ and ternary designs with block size an odd number greater than three, $\Lambda=2$, and $\Lambda(V-1)/(K-1) - \Lambda(V-1)/(K-2) \leq 2$ are also described.

1. INTRODUCTION.

In the current literature, a ***balanced ternary design*** (BTD) with parameters (V,B,R,K,Λ) is defined as an arrangement of V treatments in B blocks (i.e. multisets), each of size K, such that every treatment occurs 0, 1, or 2 times in a block. Every treatment occurs R times in the design and every pair of distinct treatments occurs Λ times in the design. This current definition [1,2] differs somewhat from the definition formulated by Tocher in 1952 [4]. Tocher in his definition did not restrict R, the replication number, to a constant, but rather let it take on multiple values in the range $\Lambda(V-1)/(K-1)$ to $\Lambda(V-1)/(K-2)$.

Using a counting argument and either of the above definitions, it can be shown [3] that the replication number of a treatment x, denoted R_x, is $(\Lambda(V-1)+2B_x)/(K-1)$ or equivalently $(\Lambda(V-1)-b_x)/(K-2)$. B_x denotes the number of blocks x appears in doubly and b_x denotes the number of blocks x appears in singly in the design. These expressions for R_x can be used to show that any replication number naturally falls into the range of values between and including $\Lambda(V-1)/(K-1)$, henceforth denoted L_R, and $\Lambda(V-1)/(K-2)$, denoted U_R.

Francel and Sarvate [3] have analyzed ternary designs where the replication numbers of the design are the boundary values L_R and U_R. In this paper, we investigate the other end of the spectrum. We analyze the situations where the replication numbers of the design cover all or all but one of the values in the range L_R to U_R.

In Section 2, we give several parametric relationships that prove useful for analyzing designs with multiple replication numbers. In Sections 3 and 4, we look at designs with block size three. In Section 5 and 6, we look at designs with even block size. In Section 7, we begin the analysis of designs with odd block size.

2. SOME USEFUL PARAMETRIC RELATIONSHIPS.

The number of distinct replication numbers for a design cannot be greater than the number of treatments in the design. Thus, if D_R represents the number of values in the range L_R to U_R that are not used as replication numbers for a design $D(V,K,\Lambda)$, then the number of treatments in D will be greater than or equal to $U_R-L_R+1-D_R$. This yields:

$$\Lambda \leq (K-1)(K-2) + D_R(K-1)(K-2)/(V-1). \qquad \textbf{(2.1)}$$

Inequality 2.1 can be used to calculate the results displayed in Table 1.

K	Number of possible replication numbers not used.	Possible Λ values
3	0	2
3	1	2
4	0	2,....,6
4	1	2,...., 6+6/(V-1)

Table 1

In the sections that follow, we determine possible V for each set of parameters shown in Table 1. Throughout our discussion, we use $TD(V,K,\Lambda)$ or simply TD to denote a ternary design in the Tocher sense.

Recall that for a design D and a treatment x, B_x denotes the number of blocks x appears in doubly and b_x denotes the number of blocks x appears in singly in D. In a BTD with a fixed replication number, the numbers B_x and b_x are independent of x. In the case of TDs with multiple replication numbers, a corresponding result occurs. The number of times a treatment x appears singly and the number of times a treatment appears doubly in a design is dependent only on the value of R_x.

<u>Lemma 2.2</u>: Let x be a treatment in a $TD(V,K,\Lambda)$ with replication number $R_x = L_R + t$ for $0 \leq t \leq U_R-L_R$, then

(a) $B_x = t(K-1)/2$, and

(b) $b_x = L_R - t(K-2)$.

Proof: The above expressions for B_x and b_x can be derived using the equations $L_R = \Lambda(V-1)/(K-1)$, $R_x = L_R + t = b_x + 2B_x$, and $b_x(K-1) + 2B_x(K-2) = \Lambda(V-1)$. []

Corollary 2.3: Let x be a treatment in a TD(V,K,Λ) with replication number $R_x = L_R + t$ for $0 \le t \le U_R - L_R$. If t is odd, then K-1 is even (i.e. K is odd).

Proof: Follows from Lemma 2.2 (a). []

Corollary 2.3 can also be stated as:

Corollary 2.4: If D is a TD(V,K,Λ) with K even, then L_R+1, L_R+3, ... are not replication numbers of D.

The values of B_x and b_x for the boundary values of t will be used in the work of Sections 3-7 so we record them here in the form of a lemma.

Lemma 2.5: Assume x is a treatment in a TD(V,K, Λ) with replication number $R_x = L_R + t$, then

(a) $t = 0$ iff $R_x = L_R = b_x$ iff $B_x = 0$, and

(b) $t = U_R - L_R$ iff $R_x = 2B_x = U_R$ iff $b_x = 0$.

We will also use the facts listed in Corollary 2.6 in our work. They are immediate consequences of Lemma 2.5 (b).

Corollary 2.6: Assume x is a treatment in a TD(V,K, Λ) with replication number $R_x = L_R + t$. If $t = U_R - L_R$, then R_x, U_R, and Λ are even.

3. K=3 AND D_R = 0.

In view of Equation 2.1, the only Λ value we need to consider for designs with block size three and D_R = 0 or 1 is two.

A TD where all values in the range L_R to U_R are used as replication numbers of the design is called a ***full ternary design*** (FTD). Francel and Sarvate [3] have given the following simple construction for a class of FTDs with parameters (V ≥ 2,K=3,Λ=2).

Construction 3.1: [3] For treatments 1,2,...,V, consider all V(V-1)/2 pairs (a,b) where a < b. For each such pair construct the block {a,a,b}. These blocks form an FTD.

As we show below, these are the only FTDs with block size three.

Lemma 3.2: For each set of parameters ($V \geq 2, K=3, \Lambda=2$) there exists exactly one FTD.

Proof: Let D be an FTD with parameters ($V \geq 2, K=3, \Lambda=2$). Since D is a FTD, there exists some treatment x in D with $R_x = U_R$. This implies x must appear doubly with every other treatment (Lemma 2.5 b). Thus for some ordering of the treatments, a subset of the design D looks like $\{\{x_1,x_1,x_2\},\{x_1,x_1,x_3\},\ldots,\{x_1,x_1,x_V\}\}$.

Similarly, since there exists a treatment x with $R_x = U_R-1$, the 2nd largest possible replication value, there exists x in D such that x appears doubly with every treatment but one (Lemma 2.2 b). Using this fact together with the information from above, we see that for some ordering of the treatments a subset of the design looks like $\{\{x_1,x_1,x_2\},\{x_1,x_1,x_3\},\ldots,\{x_1,x_1,x_V\};\ \{x_2,x_2,x_3\},\{x_2,x_2,x_4\},\ldots,\{x_2,x_2,x_V\}\}$.

Continuing this argument for V-2 more steps yields the design of Construction 3.1. []

Using Inequality 2.1, Construction 3.1, and Lemma 3.2 we get:

Theorem 3.3: FTDs with parameters ($V \geq 2, K=3, \Lambda$) exist if and only if $\Lambda = 2$. For a given set of parameters ($V \geq 2, K=3, \Lambda=2$), the existing FTD is unique.

As we will see below, such a uniqueness result is not possible when D_R is one.

4. K=3 AND D_R = 1.

The argument used in the proof of Lemma 3.2 can also be used to prove the following result.

Lemma 4.1: IF D is a TD($V, K=3, \Lambda=2$) such that each value of the decreasing sequence $U_R, U_R-1, \ldots, U_R-(j-1)$, $1 \leq j \leq V$ appears as a replication number for some treatment of D, then for some ordering of the treatments of D, the subset $\{\{x_1,x_1,x_2\},\{x_1,x_1,x_3\},\ldots,\{x_1,x_1,x_V\};\ \{x_2,x_2,x_3\},\{x_2,x_2,x_4\},\ldots,\{x_2,x_2,x_V\};\ \ldots;\ \{x_j,x_j,x_{j+1}\},\{x_j,x_j,x_{j+2}\},\ldots,\{x_j,x_j,x_V\}\}$ is a subset of the design D.

Corollary 4.2: Each value of the sequence of Lemma 4.1 appears as the replication number of only one treatment in the TD($V, K=3, \Lambda=2$).

Corollary 4.3: If D is a TD(V,K=3, Λ=2) with D_R =1, the value from the range L_R to U_R that does not appear as a replication number for D is greater than the value that appears as the replication number for two treatments in D.

In Corollary 4.3, we are using the fact that there are exactly V values that can act as replication numbers of a TD(V,K=3, Λ=2). When this is true, if one of these values is not used as a replication number, then there must be another that acts as the replication number for two distinct treatments.

We present below two construction methods for generating classes of TD(V,K=3,Λ=2) with D_R=1. In our first construction method, the value in the range L_R to U_R that does not appear as a replication number of the design is always U_R. One such example with parameters (V=4,K=3,Λ=2) is given below. In this example and in all future design examples, each column represents a block of the design.

1	1	1	3	4
2	3	3	2	4
2	4	4	2	2

For the above design, the range of possible values for replication numbers is 3 to 6. Note the treatments 1 and 3 appear three times in the design, 4 appears four times in the design, and 2 appears five times in the design. We use the above design example together with the following construction to show that ternary designs of this type exist for all V.

Construction 4.4: Assume D is a TD(V,K=3,Λ=2) where the replication numbers of D cover the set {V-1,...,2V-3}. Let x_1,...,x_V denote the design treatments and assume that x_V has the same replication number as one of the other treatments in the design.

Define D' to be the blocks of D with extra blocks {y,x_V,x_V} and {y,y,x_i} for i=1,...V-1. It is straight forward to see that D' is a ternary design with parameters (V+1,K=3,Λ=2). If R_i is the replication number of x_i in D, then in D' the replication number of x_i, for i=1,...,V-1, is R_i+1. The replication number of x_V is R_V+2. Since by assumption $R_V \leq$ 2V-3, $R_V+2 \leq$ 2V-1. Thus, the replication numbers for D' cover the set {V-1,...,2V-1}.

Now a straightforward induction proof will give:

Theorem 4.5: Designs with parameters (V≥4,K=3,Λ=2) and replication numbers that cover the range V-1 to 2V-3 exist for all V ≥ 4.

Our next construction method uses the FTDs of Construction 3.1. It is more general than Construction 4.4 in the sense that it allows the choice, with some restriction, of the value that doesn't appear as a replication number in the design. We present this construction as part of the following theorem.

Theorem 4.6: For parameters $(V\geq 4, K=3, \Lambda=2)$ and $j \geq 3$, there exists a BTD with the stated parameters and replication numbers all values in the range V-1 to 2V-2 except v-1+j.

Proof: Let D be the FTD with parameters $(V\geq 4, K=3, \Lambda=2)$ and let $j \geq 3$. Assume that x_1 is the treatment of D with replication number v-1+j. Since $j \geq 3$, there exist treatments x_2, x_3, and x_4 such that $\{x_1,x_1,x_2\}$, $\{x_1,x_1,x_3\}$, $\{x_1,x_1,x_4\}$, $\{x_2,x_2,x_3\}$, and $\{x_3,x_3,x_4\}$ are blocks in D. Remove these blocks from D and replace them with the blocks $\{x_1,x_2,x_2\}$, $\{x_1,x_3,x_4\}$, $\{x_1,x_3,x_4\}$, and $\{x_2,x_3,x_3\}$. It is clear that the balance in the new design is the same as the balance in the original design. The only treatment whose replication number has changed is x_1. It will go from V-1+j to V-1+j-3. []

If we use Construction 4.4 to build a TD(5,3,2) with replication numbers 4,5,6, and 7, we get the following design.

1	1	1	3	4	5	5	5	5
2	3	3	2	4	5	5	1	5
2	4	4	2	2	2	3	1	4

If we use the construction method of Theorem 4.6 to build a TD(5,3,2) with replication numbers 4,5,6, and 7, we get the following design.

1	1	1	2	1	2	2	3	4
2	3	3	3	1	2	2	3	4
2	4	4	3	5	4	5	5	5

These two designs are not isomorphic. In the first design, the treatment 3 is the only treatment that appears four times in the design. Twice it appears in blocks with one other treatment, and twice it appears in blocks with two other treatments. In the second design, the treatment 5 is the only treatment that appears four times in the design. It appears in four blocks that each contain only one other treatment. From this we can conclude that the designs produced by Construction 4.4 and Theorem 4.6 are not the same.

Neither of the above construction methods produced designs where the value not used as a replication number was one of the smallest three numbers in the range of possible replication values. We show below that these designs do not exist.

Lemma 4.7: For parameters ($V \geq 4, K=3, \Lambda=2$) and $j < 3$, there does not exist a TD with the given parameters and replication numbers all values in the range L_R to U_R except $V-1+j$.

Proof: Assume that every value in the sequence L_R to U_R except one of the smallest three is a replication number for some treatment in a ternary design D. Lemma 4.1 tells us exactly what the blocks in the design look like except for those blocks that are constructed from only the treatments x_{V-2}, x_{V-1}, and x_V, where $x_1, x_2, \ldots, x_V$ is the ordering of the treatments as given in Theorem 4.1. One can argue that there is no way to extend the known collection of blocks to a TD so that only one of the three values L_R, L_R+1, or L_R+2 does not appear as a replication number for some treatment. []

Inequality 2.1, Theorem 4.6, and Lemma 4.7 together give:

Theorem 4.8: A TD($V \geq 4, K=3, \Lambda$) with $D_R = 1$ exists, if and only if $\Lambda = 2$ and the value from L_R to U_R not used as a replication number is larger than L_R+2.

5. K EVEN and $D_R = 0$.

In this section and the next, we restrict ourselves to looking at designs with even block size. The results we show for designs with K even are surprising. As we see below, FTDs with K even do not exist and TDs with K even and $D_R = 1$ exist only under very limited conditions.

Lemma 5.1: If D(V,K,Λ) is a design with K even, and if R_x is even for some treatment x in D, then R_x is even for every treatment x in D.

Proof: Assume that D(V,K,Λ) is a design with R_x even for some treatment x in D. Recall $R_x = (\Lambda(V-1)+2B_x)/(K-1)$. Since R_x is even for some treatment x in D, $\Lambda(V-1)$ is even. Since K-1 is odd, the result follows. []

Theorem 5.2: No FTDs with K even exist.

Proof: This is immediate from Lemma 5.1 since in an FTD replication numbers will be both even and odd. []

6. K EVEN and $D_R = 1$.

Lemma 5.1 allows us to show that TDs with K even and $D_R=1$ don't exist unless $U_R-L_R = 1$.

Theorem 6.1: No design D(V,K, Λ) with K even, $D_R=1$, and $U_R-L_R \geq 3$ exists.

Proof: With at least four values in the range L_R to U_R and with $D_R=1$, at least one odd and one even value would have to appear as replication numbers for the design. Thus, the result follows from Lemma 5.1. []

Theorem 6.1 does not cover two cases. The case where the range L_R to U_R contains exactly three values and the case where the range contains exactly two values.

Case 1 ($U_R-L_R = 1$): If there are only two values in the range L_R to U_R, and if only L_R is used as a replication number, the design is a BIBD (Lemma 2.5 a), as every treatment appears singly. If there are only two values in the range L_R to U_R, and if only U_R is used as a replication number, then every treatment in the design appears only doubly (Lemma 2.5 b). A design where treatments appear only doubly corresponds in a natural way to a BIBD. Namely, replace each double appearance of a treatment with a single appearance of the same treatment).

Below we show that the BIBDs of the above paragraph cannot exist. To do this we will use the following well known parametric relationships for BIBD(v,b,r,k,λ): (1) $v \leq b$, (2) $bk=rv$, and (3) $\lambda(v-1) = r(k-1)$.

Note, the range L_R to U_R contains exactly two values if and only if $\Lambda(V-1) = (K-1)(K-2)$.

Lemma 6.2: No BIBD with parameters (V,B, Λ(V-1)/(K-1),K, Λ) exists if $\Lambda(V-1) = (K-1)(K-2)$.

Proof: Assume $\Lambda(V-1) = (K-1)(K-2)$ and a BIBD(V,B,Λ(V-1)/(K-1),K, Λ) exists. Since $\Lambda(V-1) = (K-1)(K-2)$, we can use (3) to conclude that $R = K-2$. Thus, using (2) we see that $V = BK/(K-2)$ which implies that $V > B$ a contradiction to (1). []

Lemma 6.3: No BIBD with parameters (V,B, Λ(V-1)/2(K-2),K/2, Λ/4) exists if $\Lambda(V-1) = (K-1)(K-2)$.

Proof: Assume $\Lambda(V-1) = (K-1)(K-2)$ and a BIBD$(V,B, \Lambda(V-1)/2(K-2),K/2, \Lambda/4)$ exists. Using (2) and the fact that $\Lambda(V-1) = (K-1)(K-2)$, we get $V = BK/(K-1)$ which implies that $V > B$, a contradiction to (1). []

Theorem 6.4: No design $D(V,K, \Lambda)$ with K even, $D_R=1$, and $U_R-L_R=1$ exists.

Proof: Lemmas 6.2 and 6.3 and the discussion above Lemma 6.2. []

Case 2 (U_R-L_R = 2): Here we look at the case where $U_R-L_R = 2$. We use Corollaries 2.4 and 2.6 to conclude that the middle value in the range must be odd and it must be the value that is not used as a replication number. Thus, this case reduces to the case where the two bound values L_R and U_R are the design replication numbers. This case is discussed by Francel and Sarvate [3]. We note that designs of this type exist. The TD($V=4,K=4, \Lambda=4$) shown below is such a design. In this design $L_R=4$, $U_R=6$, and the design replication numbers are 4, and 6.

1	1	1	2	2
1	1	1	2	2
2	3	3	3	3
2	4	4	4	4

The results of this section summarize as:

Theorem 6.5: If a TD with parameters $(V,K=2K', \Lambda)$ and $D_R=1$ exists, then $U_R-L_R = 2$.

The conditions of Theorem 6.5 are not sufficient. For example, as part of a more general result in Section 7 we will prove the following:

Theorem 6.6: If a TD with parameters $(V,K=2K', \Lambda=2)$, $D_R=1$, and $U_R-L_R = 2$ exists, then $V=73$ and $K=10$.

The question of exactly which conditions are both necessary and sufficient when K is even and $D_R=1$ remains open.

7. K ODD.

Sections 3 and 4 give us complete results for the cases $K=3$ and $D_R= 0$ or 1. The complexity and results of the situations for K odd but greater than three, and $D_R = 0$ or 1 however appear to model more closely the K even case than the K=3 case. To give a flavor of the odd greater than three block size case, we give below some general nonexistence results and analyze part of the K odd, and $\Lambda=2$ case.

The parametric relationships $\Lambda(V-1) = (U_R-L_R)(K-1)(K-2)$, $U_R=(U_R-L_R)(K-1)$, and $L_R=(U_R-L_R)(K-2)$ lead to the restrictions shown in Table 2:

V	K	Λ	Restrictions
odd	odd	odd	$b_x \geq 1$, $\forall$ x in D
even	odd	odd	nonexistence
odd	odd	even	U_R-L_R odd
even	odd	even	U_R-L_R even

Table 2

These restrictions give us the following:

<u>Theorem 7.1</u>: If a TD(V,K, Λ) with K odd, Λ odd, and $D_R=1$ exists, then V is odd and the value from the range L_R to U_R that is not used as a replication number for the design is U_R.

<u>Theorem 7.2</u>: TDs with parameters (V,K, Λ) do not exist for V even, K odd, and Λ odd.

Also, the above together with Theorem 5.2 gives us:

<u>Theorem 7.3</u>: FTDs with parameters (V,K, Λ) do not exist for odd Λ.

Next, assume for a TD(V,K, $\Lambda=2$), $D_R = 0$ or 1. Also assume L_R and U_R are the only replication numbers of the design.

Since by assumption $D_R = 0$ or 1 and L_R and U_R are the only replication numbers of the design, we know $U_R-L_R=s$ where s = 1 or 2 (i.e. for s = 1 or 2, $2(V-1) = s(K-1)(K-2)$). Since $\Lambda=2$, only one treatment can have replication number U_R. Thus, since L_R and U_R are the only replication numbers for the design, the other V-1 treatments will have replication number L_R and $U_R+(V-1)L_R = BK$. This yields:

$$K(s^2K^2 - 5s^2K + 8s^2 + 2s - 2B) = 2s(2s+1) \qquad \textbf{(7.4)}$$

Case 1 (U_R-L_R=1): If $U_R-L_R=1$, then Equation 7.4 becomes: $K(K^2-5K+10-2B) = 6$, which implies K divides 6. Hence, since K is odd and greater than three, we can conclude:

Lemma 7.5: FTDs with parameters ($V,K,\Lambda=2$) and U_R-L_R=1 do not exist for K greater than three and odd.

Proof: K greater than three and odd implies K does not divide 6. []

Recall from Theorem 5.2 that FTDs do not exist when K is even. Thus, Theorem 3.3, 5.2 and 7.2 together give:

Theorem 7.6: FTDs with parameters ($V\geq2,K,\Lambda=2$) and U_R-L_R=1 exist if and only if K=3 for a given set of parameters ($V\geq2,K=3,\Lambda=2$), the existing FTD is unique.

Case 2 (U_R-L_R = 2): If U_R-L_R=2, then Equation 7.1 becomes:
$K(2K^2-10K+10-B) = 10$ which implies K divides 10. Hence K is 5 or 10.
If K=5, then V=13, since V-1=(K-1)(K-2). Such a design exists. The example below is one such design. The replication numbers for the design are 6 and 8. The third possible replication number, 7, is not used in the design.

1	1	1	1	2	2	3	5	5	6	8	8	9	11	11	12
1	1	1	1	3	4	4	6	7	7	9	10	10	12	13	13
2	5	8	11	5	6	7	2	3	4	2	3	4	2	3	4
3	6	9	12	8	9	10	10	9	8	7	6	5	7	6	5
4	7	10	13	11	12	14	13	12	11	13	12	11	10	9	8

Theorem 7.7: TDs with parameters ($V,K,\Lambda=2$), K odd, D_R=1, and U_R-L_R=2 exist if and only if V=13 and K=5.

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References

1. Billington, Elizabeth J., "Balanced n-ary designs: A combinatorial survey and some new results," *Ars Combinatoria*, **17A** (1984), 37-72.

2. Billington, Elizabeth J., "Designs with repeated elements in blocks: A survey and some recent results," *Congressus Numerantium*, **68** (1989), 123-146.

3. Francel, M. and Sarvate, D., "Ternary Designs With More Than One Replication Number," *Congressus Numerantium*, **101** (1994), 17-25.

4. Tocher, K.D., "The design and analysis of block experiment, "J. Royal *Statist. Soc.* Ser B, **14** (1952), 45-100.

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