On Packing Designs with Block Size 5 and Indices 3 and 5

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<u>Abstract</u> Let V be a finite set of order υ. A $(\upsilon,\kappa,\lambda)$ packing design of index λ and block size κ is a collection of κ-element subsets, called blocks, such that every 2-subset of V occurs in at most λ blocks. The packing problem is to determine the maximum number of blocks, $\sigma(\upsilon,\kappa,\lambda)$, in a packing design. It is well known that $\sigma(\upsilon,\kappa,\lambda) \le \left[\frac{v}{\kappa}\left[\frac{v-1}{\kappa-1}\lambda\right]\right] = \psi(v,\kappa,\lambda)$, where $[x]$ is the largest integer satisfying $x \ge [x]$. It is shown here that $\sigma(\upsilon,5,3) = \psi(\upsilon,5,3)$ for all $\upsilon \equiv 3 \pmod 4$ and $\sigma(\upsilon,5,5) = \psi(\upsilon,5,5)$ for all positive integers $\upsilon \ge 5$ with the possible exceptions of $\upsilon = 28, 32, 34$.

1. Introduction

A $(\upsilon,\kappa,\lambda)$ packing design (or respectively covering design) of order υ, block size κ and index λ is a collection β of κ-element subsets, called blocks, of a υ-set V such that every 2-subset of V occurs in at most (at least) λ blocks.

Let $\sigma(\upsilon,\kappa,\lambda)$ denote the maximum number of blocks in a $(\upsilon,\kappa,\lambda)$ packing design; and $\alpha(\upsilon,\kappa,\lambda)$ denote the minimum number of blocks in a $(\upsilon,\kappa,\lambda)$ covering design. A $(\upsilon,\kappa,\lambda)$ packing design with $|\beta| = \sigma(\upsilon,\kappa,\lambda)$ will be called a maximum packing design. Similarly, a $(\upsilon,\kappa,\lambda)$ covering design with $|\beta| = \alpha(\upsilon,\kappa,\lambda)$ is called a minimum covering design. It is well known [23] that

$$\sigma(v,\kappa,\lambda) \le \left[\frac{v}{\kappa}\left[\frac{v-1}{\kappa-1}\lambda\right]\right] = \psi(v,\kappa,\lambda) \text{ and } \alpha(v,\kappa,\lambda) \ge \left\lceil\frac{v}{\kappa}\left\lceil\frac{v-1}{\kappa-1}\right\rceil\right\rceil = \phi(v,\kappa,\lambda)$$

where $[x]$ is the largest integer satisfying $[x] \le x$ and $\lceil x\rceil$ is the smallest integer satisfying $x \le \lceil x\rceil$. When $\sigma(\upsilon,\kappa,\lambda) = \psi(\upsilon,\kappa,\lambda)$ the $(\upsilon,\kappa,\lambda)$ packing design is called optimal packing design. Similarly when $\alpha(\upsilon,\kappa,\lambda) = \phi(\upsilon,\kappa,\lambda)$ the $(\upsilon,\kappa,\lambda)$ covering design is called minimal covering design.

Many researchers have been involved in determining the packing number

$\sigma(v,\kappa,\lambda)$ known up to date. The following theorem summarizes what is known about packing pairs by quintuples.

Theorem 1.1 Let $v \geq 5$ be a positive integer. Then

1) $\sigma(v,5,1) = \psi(v,5,1)$ for $v \equiv 3 \pmod{20}$ and $v \equiv 0 \pmod 4$ $v \neq 12, 16$ with the possible exception of v = 32, 48, 52, 72, 132, 152, 172, 232, 243, 252, 272, 332, 352, 432 [16] [18] [26], and $\sigma(12,5,1) = \psi(12,5,1) - 1$, $\sigma(16,5,1) = \psi(16,5,1) - 1$ [16].

2) $\sigma(v,5,2) = \psi(v,5,2)$ for all positive even integers v, [5] and $\sigma(v,5,2) = \psi(v,5,2) - e$ where $e = 1$ if $v \equiv 7$ or $9 \pmod{10}$ or $v = 13$ with the possible exception of v = 15, 19, 27 and $e = 0$ if $v \equiv 1, 3$ or $5 \pmod{10}$ $v \neq 13,15$ [6,7].

3) (a) $\sigma(v,5,3) = \psi(v,5,3)$ for all positive integers v, $v \not\equiv 0 \pmod 4$ with the possible exception of v = 17, 19, 29, 33, 38, 49 [8,13].

(b) $\sigma(v,5,3) = \psi(v,5,3)$ for all positive integers $v \equiv 0 \pmod 4$ $v \leq 96$ with the possible exception of v = 20, 28, 32, 36, 56, [9].

4) $\sigma(v,5,4) = \psi(v,5,4)$ for all positive integers v, $v \neq 7$ and $\sigma(7,5,4) = \psi(7,5,4) - 1$ [12].

5) $\sigma(v,5,6) = \psi(v,5,6)$ for all positive integers v, with the possible exception of $v = 43$ [13].

6) $\sigma(v,5,\lambda) = \psi(v,5,\lambda) - e$ for all positive integers v and λ = 8, 12, 16 [11] with few possible exceptions where $e = 1$ if $\lambda(v-1) \equiv 0 \pmod 4$ and $\frac{\lambda v(v-1)}{4} \equiv 1 \pmod 5$ and $e = 0$ otherwise.

Furthermore, these few possible exceptions were removed later, in an unpublished paper, by Shalaby [24].

Our interest here is in the case $\kappa = 5$ and $\lambda = 3, 5$. Our goal is to prove the following:

Theorem 1.2 Let $v \geq 5$ be a positive integer. Then $\sigma(v,5,3) = \psi(v,5,3)$ for all $v \equiv 3 \pmod 4$ and $\sigma(v,5,5) = \psi(v,5,5)$ for all positive integers $v \geq 5$ with the possible exception of v = 28, 32, 34.

2. Recursive Constructions

In order to describe our recursive constructions we require several other types of combinatorial design. A balanced incomplete block design, $B[v,\kappa,\lambda]$, is a (v,κ,λ) packing design where every 2-subset of points is contained in precisely λ blocks. If a $B[v,\kappa,\lambda]$ exists then it is clear that $\sigma(v,\kappa,\lambda) = \lambda v(v-1)/\kappa(\kappa-1) = \psi(v,\kappa,\lambda)$ and Hanani [16] has proved the following existence theorem for $B[v,5,\lambda]$.

<u>Lemma 2.1</u> Necessary and sufficient conditions for the existence of a $B[v,5,\lambda]$ are that $\lambda(v-1) \equiv 0 \pmod 4$ and $\lambda v(v-1) \equiv 0 \pmod{20}$ and $(v,\lambda) \neq (15, 2)$.

<u>Corollary</u> $\sigma(v,5,5) = \psi(v,5,5)$ for all positive integers v where $v \equiv 1 \pmod 4$.

A (v,κ,λ) packing design with a hole of size h is a triple (V,H,β) where V is a v-set, H is a subset of V of cardinality h, and β is a collection of κ-element subsets, called blocks, of V such that

1) no 2-subset of H appears in any block;
2) every other 2-subset of V appears in at most λ blocks;
3) $|\beta| = \psi(v,\kappa,\lambda) - \psi(h,\kappa,\lambda)$.

It is clear that if there exists a (v,κ,λ) packing design with a hole of size h and $\sigma(h,\kappa,\lambda) = \psi(h,\kappa,\lambda)$ then $\sigma(v,\kappa,\lambda) = \psi(v,\kappa,\lambda)$.

Let κ, λ and v be positive integers and M be a set of positive integers. A group divisible design $GD[\kappa,\lambda,M,v]$ is a triple (V,β,γ) where V is a set of points with $|V| = v$, and $\gamma = \{G_1,\ldots, G_n\}$ is a partition of V into n sets called groups. The collection β consists of κ-subsets of V, called blocks, with the following properties

1) $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_i \in \gamma$;
2) $|G_i| \in M$ for all $G_i \in \gamma$;
3) every 2-subset $\{x,y\}$ of V such that x and y belong to distinct groups is contained in exactly λ blocks.

If $M = \{m\}$ then the group divisible design is denoted by $GD[\kappa,\lambda,m,v]$.

A $GD[\kappa,\lambda,m,\kappa m]$ is called a transversal design and denoted by $T[\kappa,\lambda,m]$. It is well known that a $T[\kappa,1,m]$ is equivalent to $\kappa-2$ mutually orthogonal Latin squares of side m.

In the sequel we shall use the following existence theorem for transversal designs. The proof of this result may be found in [1], [2], [14], [15], [18], [22], [24].

Theorem 2.1 There exists a T[6,1,m] for all positive integers m with the exception of m ∈ {2,3,4,6} and the possible exception of m ∈ {10, 14, 18, 22, 26, 34, 42}.

Theorem 2.2 If there exists a GD[6,λ,5,5n] and a (20+h,5,λ) packing design with a hole of size h then there exists a (20(n-1)+4u+h,5,λ) packing design with a hole of size 4u + h where 0 ≤ u ≤ 5.

Proof Take a GD[6,λ,5,5n] and delete 5-u points from the last group. Inflate this design by a factor of 4. On the blocks of size 5 and 6 construct a GD[5,1,4,20] and a GD[5,1,4,24] respectively, lemma 2.1. Add h points to the groups and on the first n-1 groups construct a (20+h,5,λ) packing design with a hole of size h, and take the h points with the last group to be the hole of size 4u+h.

It is clear to apply the above theorem we require the existence of a GD[6,λ,5,5n]. Our authority for that is the following lemma of Hanani [18, p.286].

Lemma 2.2 There exists a GD[6,λ,5,35] for λ = 3, 5.

If in the definition of GD[κ,λ,m,υ] (similarly T[κ,λ,m]) condition (3) is changed to be read as (3) every 2-subset {x,y} of V such that x and y are neither in the same group (column) nor in the same row is contained in exactly λ blocks of β and no block contains more than one point from the same row. Then the resultant design is called a modified group divisible design (modified transversal design) and is denoted by MGD[κ,λ,m,υ] (MT[κ,λ,m]). (We may look at the points of MGD[κ,λ,m,υ] as the points of a matrix and then the groups of MGD[κ,λ,m,υ] are precisely the columns of the matrix).

A resolvable modified group divisible design, RMGD[κ,λ,m,υ], is a modified group divisible design the blocks of which can be partitioned into parallel classes.

It is clear that a RMGD[5,1,5,5m] is the same as RT[5,1,m] with one parallel class of blocks singled out, and since RT[5,1,m] is equivalent to T[6,1,m] we have the following

Theorem 2.3 There exists a RMGD[5,1,5,5m] for all positive integers m with the exception of m ∈ {2,3,4,6} and the possible exception of m ∈ {10, 14, 18, 22, 26,34, 42}.

The next two theorems are in the form most useful to us.

Theorem 2.4 [3] If there exists a RMGD[5,1,5,5m] and a GD[5,λ,{4,s*}, 4m+s], where * means there is exactly one group of size s, and there exists a (20+h,5,λ) packing design with a hole of size h then there exists a (20m+4u+h+s,5,λ) packing design with a hole of size 4u+h+s where $0 \le u \le m-1$.

Theorem 2.5 If there exists (1) a RMGD[5,1,5,5m] (2) a GD[5,5,4,4m] (3) a (24,5,5) packing design with a hole of size 4 (4) $\sigma(24,5,5) = \psi(24,5,5)$. Then $\sigma(20m+4,5,5) = \psi(20m+4,5,5)$.

Proof Inflate a RMGD[5,1,5,5m] by a factor of 4, that is, replace the blocks of size 5 by the blocks of GD[5,5,4,20]. On the rows (which are blocks of size m) construct a GD[5,5,4,4m]. Finally add 4 points to the groups and on the first (m-1) groups construct a (24,5,5) packing design with a hole of size 4 and on the last group construct a (24,5,5) optimal packing design.

It is clear that the application of the above theorem requires the existence of a GD[5,1,{4,s*}, 4m+s]. The following theorem is most useful to us. For the proof of the first part see [3] and for the proof of the second part see [17].

Theorem 2.6 (i) There exists a GD[5,1,{4,s*},4m+s] where s = 0 if $m \equiv 1 \pmod 5$, s = 4 if $m \equiv 0$ or $4 \pmod 5$ and $s = \frac{4(m-1)}{3}$ if $m \equiv 1 \pmod 3$.

(ii) There exists a GD[5,1,{4,8*}, 4m+8] where $m \equiv 0$ or $2 \pmod 5$, $m \ge 7$ with the possible exception of m = 10.

In the case m = 7, 8, 13 the following lemma is most useful to us.

Lemma 2.3 There exists a GD[5,5,4,v] where v = 28, 32, 52

Proof For v = 28 let X = Z_{28}. The groups are < 0 7 14 21 > + i, i $\in Z_7$ and the blocks are the following:

< 0 1 3 9 13 > (mod 28), < 0 4 9 15 20 > (mod 28), < 0 1 2 3 4 > (mod 28)

<0 3 9 13 19 > (mod 28), < 0 2 8 13 18 > (mod 28), < 0 3 11 15 20 > (mod 28).

For a GD[5,5,4,32] let $X = Z_{32}$. The groups are < 0 8 16 24 > + i, $i \in Z_8$

Blocks:

< 0 2 7 11 20 > (mod 32) < 0 1 2 4 11 > (mod 32) < 0 3 7 17 22 > (mod 32)

< 0 5 11 17 23 > (mod 32) < 0 1 2 4 13 > (mod 32) < 0 1 5 11 18 > (mod 32)

< 0 3 6 13 18 > (mod 32)

For a GD[5,5,4,52], since there exists a B[13,5,5] and a GD[5,1,4,20] it follows, [16 lemma 2.16], that there exists a GD[5,5,4,52].

The set of blocks $\langle \kappa\ \ \kappa+m\ \ \kappa+n\ \ \kappa+j\ \ f(\kappa)\rangle$ (mod v) for $\kappa = 0,\ldots, v-1$ where $f(\kappa) = a$ if κ is even and $f(\kappa) = b$ if κ is odd will be denoted by $\langle 0\ m\ n\ j\rangle \cup \{a,b\}$, and the set of blocks $\langle \kappa\ \ \kappa+m\ \ \kappa+n\ \ \kappa+j\ \ f(\kappa)\rangle$ (mod v) for $\kappa = 0,\ldots, v-1$ where $f(\kappa) = h_i$ if $\kappa \equiv i \pmod 4$ is denoted by $\langle 0\ m\ n\ j\rangle \cup \{h_i\}^4_{i=1}$. Similarly, the set of blocks $\langle (0,\kappa)\ \ (0,\kappa+m)\ \ (1,\kappa+n)\ \ (1,\kappa+j)\ \ f(\kappa)\rangle$ mod $(-,v)$ for $\kappa=0,\ldots, v-1$ where $f(\kappa) = a$ if κ is even and $f(\kappa) = b$ if κ is odd is denoted by $\langle (0,0)\ \ (0,m)\ \ (1,n)\ \ (1,j)\rangle \cup \{a,b\}$.

3. The Structure of Packing and Covering Designs

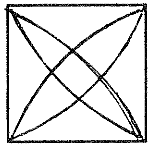
Let (V,β) be a (v,κ,λ) packing design, and for each 2-subset $e = \{x,y\}$ of V define m(e) to be the number of blocks in β which contain e. Note that by the definition of a packing design we have $m(e) \le \lambda$ for all e.

The complement of (V,β), denoted by $C(V,\beta)$ is defined to be the graph with vertex set V and edges e occurring with multiplicity $\lambda - m(e)$ for all e. The number of edges (counting multiplicities) in $C(V,\beta)$ is given by $\lambda\binom{v}{2} - |\beta|\binom{\kappa}{2}$. The degree of the vertex x in $C(V,\beta)$ is $\lambda(v-1) - r_x(\kappa-1)$ where r_x is the number of blocks containing x.

In a similar way we define the excess graph of a (V,β) covering design denoted by $E(V,\beta)$, to be the graph with vertex set V and edges e occurring with multiplicity $m(e) - \lambda$ for all e. The number of edges in $E(V,\beta)$ is given by $|\beta|\binom{\kappa}{2} - \lambda\binom{v}{2}$; and the degree of each vertex is $r_x(\kappa-1) - \lambda(v-1)$ where r_x is as before.

<u>Lemma 3.1</u> Let (V,β) be a $(v,5,4)$ covering design with $|\beta| = \phi(v,\kappa,\lambda)$ then the degree of each vertex of $E(V,\beta)$ is divisible by 4 and the number of edges in the graph is 0, 6, 8 when v mod 5 $\in \{0,1\}, \{2,4\}, \{3\}$ respectively.

In the case $\upsilon \equiv 3 \pmod 5$ a particularly useful graph with 8 edges and each vertex of degree divisible by 4 is the one that consists of $\upsilon-4$ isolated vertices and the following graph on the remaining 4 vertices.



To define the complement graph of a packing design with a hole H of size h let e = {x,y} where at least one of x or y does not lie in H and let m(e) be the number of blocks in β which contain e. Then the complement graph of the packing design with a hole H of size h, denoted by $C(V\backslash H,\beta)$, is the graph with vertex set V and edges e occuring with multiplicity λ-m(e). In a similar way the excess graph, $E(V\backslash H,\beta)$, of a $(\upsilon,\kappa,\lambda)$ covering design with a hole of size h is defined.

4. Packing Designs with Index 3 and Order $\upsilon \equiv 3 \pmod 4$

<u>Lemma 4.1</u> For all $\upsilon \equiv 3 \pmod{20}$ we have $\sigma(\upsilon,5,3) = \psi(\upsilon,5,3)$. Furthermore, there exists a (23,5,3) packing design with a hole of size 3.

<u>Proof</u> For all $\upsilon \equiv 3 \pmod{20}$ a $(\upsilon,5,3)$ packing design with $\psi(\upsilon,5,3)$ blocks can be constructed as follows

1) take a $(\upsilon-1,5,2)$ optimal packing design; such design exists by [5].
2) take a $B[\upsilon+2,5,1]$, lemma 2.1, and assume in this design we have the block $< \upsilon-2\ \ \upsilon-1\ \ \upsilon\ \ \upsilon+1\ \ \upsilon+2 >$; drop this block and in all other blocks change both $\upsilon+2$ and $\upsilon+1$ to υ; which proves the first part of the lemma.

Since the (22,5,2) optimal packing design has a hole of size 2 [5, p.49] and since we droped the block < 21 22 23 24 25 > it follows that the (23,5,3) packing has a hole of size 3.

The following lemma is very useful to us.

Lemma 4.2 Let $v \equiv 6 \pmod{20}$ be a positive integer. Then there exists a $(v,5,2)$ packing design with a hole of size 6.

Proof For $v = 6, 26, 46$ see [5, p.51].

For $v = 66$ let $X = Z_{60} \cup \{\infty_i\}_{i=1}^{6}$. Then take the following blocks under the action of the group Z_{60}. $< 0\ 1\ 3\ 5\ 11 >$, $< 0\ 4\ 10\ 19\ 38 >$, $< 0\ 1\ 8\ 21\ 35 >$, $< 0\ 3\ 15\ 27\ 43 >$, $< 0\ 5\ 23\ 36 > \cup \{\infty_1, \infty_2\}$, $< 0\ 7\ 16\ 37 > \cup \{\infty_3, \infty_4\}$, $< 0\ 11\ 25\ 42 > \cup \{\infty_5, \infty_6\}$. For $v = 86$ let $X = Z_{80} \cup \{\infty_i\}_{i=1}^{6}$. On Z_{80} construct an $(80,5,1)$ minimal covering design [21], in this design each pair appears once except the pairs $\{i, i+40\}$, $i = 0, \ldots, 39$ which appear twice. Furthermore, take the following blocks under the action of the group Z_{80}.
$< 0\ 1\ 3\ 7\ 15 >$, $< 0\ 10\ 21\ 38\ 54 >$, $< 0\ 5\ 27\ 50 > \cup \{\infty_1, \infty_2\}$, $< 0\ 9\ 29\ 48 > \cup \{\infty_3, \infty_4\}$, $< 0\ 13\ 31\ 56 > \cup \{\infty_5, \infty_6\}$.

For $v \geq 106$ $v \neq 126, 146$ simple calculations show that v can be written in the form $20m+4u+h+s$ where m, u, h and s are chosen so that

1) There exists a RMGD[5,1,5,5m], theorem 2.3.
2) There exists a GD[5,2,{4,s*},4m+s], theorem 2.5.
3) $4u+h+s = 6, 26, 46, 66, 86$.
4) $0 \leq u \leq m-1$, $s \equiv 0 \pmod 4$ and $h = 6$.

Now apply theorem 2.4 with $\lambda = 2$ to get that a $(v,5,2)$ packing design with a hole of size 6, 26, 46, 66, or 86 exists and hence a $(v,5,2)$ packing design with a hole of size 6 exists.

For $v = 126, 146$ apply theorem 2.2 with $n = 7$, $\lambda = 2$, $h = 6$ and $u = 0, 5$ respectively.

Lemma 4.3 Let $v \equiv 7 \pmod{20}$ be a positive integer. Then $\sigma(v,5,3) = \psi(v,5,3)$.

Proof For $v = 7, 27, 47$ the constructions are given in the next table. In general, the construction in this table and other tables to come is as follows. Let $X = Z_{v-n} \cup H_n$ or $X = Z_2 \times Z_{\frac{v-n}{2}} \cup H_n$ where $H_n = \{h_1, \ldots, h_n\}$ is the hole. The blocks are constructed by taking the orbits of the tabulated base blocks mod $(v-n)$ or mod $(-, \frac{v-n}{2})$ respectively unless it is otherwise specified.

For all other values of v let $X = Z_{v-7} \cup H_6 \cup \{\infty_1, \infty_2, \infty_3\}$, then the construction is as follows.

1) On $Z_{v-7} \cup H_6$ construct a $(v-1,5,2)$ packing design with a hole of size 6, say, $\{h_1, \ldots, h_6\}$, lemma 4.2.

2) On $Z_{v-7} \cup H_6 \cup \{\infty_1, \infty_2, \infty_3\}$ construct a $(v+2,5,1)$ packing design with a hole of size 9, say, $\{h_1, \ldots, h_6\} \cup \{\infty_1, \infty_2, \infty_3\}$ [17]. Furthermore, replace the points ∞_2 and ∞_3 by ∞_1.

3) To the blocks obtained in (1) and (2) adjoin the following blocks

$<h_1\ h_2\ h_3\ h_4\ \infty_1>$, $<h_1\ h_2\ h_3\ h_4\ h_5>$, $<h_3\ h_4\ h_5\ h_6\ \infty_1>$, $<h_1\ h_4\ h_5\ h_6\ \infty_1>$, $<h_1\ h_2\ h_5\ h_6\ \infty_1>$.

It is readily checked that the above three steps give a $(v,5,3)$ optimal packing design.

v	Point Set	Base Blocks
7	$Z_5 \cup H_2$	$< 0\ 1\ 2\ 4 > \cup \{h_1, h_2\}$
27	$Z_2 \times Z_{12} \cup H_3$	$< (0,0)\ (0,6)\ (1,0)\ (1,6) > + (-,i),\ i \in Z_6$
		$<(0,0)\ (0,2)\ (0,6)\ (0,9)\ (1,11)>$, $<(0,0)\ (1,0)\ (1,1)\ (1,4)\ (1,6)>$
		$<(0,0)\ (0,1)\ (0,5)\ (0,10)\ (1,8)>$, $<(0,0)\ (1,3)\ (1,4)\ (1,8)\ (1,11)>$
		$< (0,0)\ (0,1)\ (1,1)\ (1,3)\ h_1 >$, $< (0,0)\ (0,4)\ (1,5)\ (1,8)\ h_3 >$
		$< (0,0)\ (0,2)\ (1,7)\ (1,9)\ h_2 >$
		$< (0,0)\ (0,1)\ (1,10)\ (1,11) > \cup \{h_1, h_2\}$.
47	$Z_{40} \cup H_7$	On $Z_{40} \cup \{h_i\}_{i=1}^{5}$ construct a B[45,5,1], lemma 2.1; drop the block
		$< h_1\ h_2\ h_3\ h_4\ h_5 >$ and take the following blocks
		$< 0\ 4\ 8\ 12\ 16 > + i,\ i \in Z_8$ twice, $< 0\ 1\ 2\ 4\ 14 >$,
		$< 0\ 4\ 9\ 19 > \cup \{h_1, h_2\}$, $< 0\ 5\ 11\ 28 > \cup \{h_3, h_4\}$,
		$<0\ 6\ 13\ 31 > \cup \{h_5, h_6\}$, $< 0\ 3\ 14\ 21 > \cup \{h_6, h_7, h_7, h_7\} >$

<u>Lemma 4.4</u> Let $v \equiv 11 \pmod{20}$ be a positive integer. Then $\sigma(v,5,3) = \psi(v,5,3)$.

<u>Proof</u> For $v = 11, 51, 91$ see the table below.

For $v = 31$ take the blocks of a $(31,5,1)$ optimal packing design [20] together with the blocks of a B[31,5,2], lemma 2.1.

For $v = 71$ take a T[5,3,14] [18] and add a new point to the groups and on each group construct a $(15,5,3)$ optimal packing design, (see lemma 4.6).

For $\upsilon \geq 111$, $\upsilon \neq 131$ simple calculations show that υ can be written in the form 20m+4u+h+s where m, u, h and s are chosen so that

1) There exists a RMGD[5,1,5,5m], theorem 2.3.
2) There exists a GD[5,3,{4,s*},4m+s], theorem 2.5.
3) 4u+h+s = 11, 31, 51, 71, 91.
4) $0 \leq u \leq m-1$, $s \equiv 0 \pmod 4$ and h = 3.

Apply theorem 2.4 with $\lambda = 3$ to get the result.

For $\upsilon = 131$ apply theorem 2.2 with n = 7, h = 3 and u = 2.

υ	Point Set	Base Blocks
11	$Z_2 x Z_5 \cup H_1$	< (0,0) (0,1) (1,0) (1,1) (1,3) >, < (0,0) (0,2) (1,0) (1,3) (1,4) < (0,0) (0,2) (0,3) (1,4) h_1 >
51	$Z_2 \times Z_{20} \cup H_{11}$	< (0,0) (0,4) (0,8) (0,12) (0,16) > + (–,i), i $\in Z_4$, twice < (1,0) (1,4) (1,8) (1,12) (1,16) > + (–,i), i $\in Z_4$ < (0,0) (0,10) (1,0) (1,10) h_{11} > + i, i $\in Z_{10}$ < (0,0)(0,10)(1,1)(1,7)(1,17) >, < (0,0)(0,3)(0,5)(0,16) > $\cup$ {h_1,h_2} <(1,0)(1,3)(1,5)(1,12)> $\cup$ {h_1,h_2}, <(0,0)(0,9)(0,15)(1,1)> $\cup$ {h_3,h_4} <(0,0)(1,0)(1,1)(1,4)> $\cup$ {h_3,h_4}, <(0,0)(0,1)(0,15)(1,0)> $\cup$ {h_5,h_6} <(0,0)(1,2)(1,5)(1,7)> $\cup$ {h_5,h_6}, <(0,0)(0,1)(0,3)(1,13)> $\cup$ {h_7,h_8} <(0,0)(1,9)(1,16)(1,18)> $\cup$ {h_7,h_8}, <(0,0)(0,1)(1,6)(1,15)> $\cup$ {h_1,h_2} <(0,0)(0,3)(1,18)(1,19)> $\cup$ {h_3,h_4}, <(0,0)(0,7)(1,9)(1,10)> $\cup$ {h_5,h_6} <(0,0)(0,7)(1,11)(1,18)> $\cup$ {h_7,h_8}, <(0,0)(0,9)(1,8)(1,13)> $\cup$ {h_9,h_{10}} < (0,0)(0,8)(1,2)(1,16) h_9 >, < (0,0)(0,2)(1,8)(1,14) h_{10} > <(0,0)(0,6)(1,3)(1,15) h_{11} >.
91	$Z_{80} \cup H_{11}$	On $Z_{80} \cup \{h_{11}\}$ construct a B[81,5,1], lemma 2.1, and take the following blocks < 0 4 12 28 40 >, < 0 5 14 32 34 >, < 0 1 7 37 61 > < 0 2 13 27 > $\cup \{h_i\}^4_{i=1}$, < 0 3 18 41 > $\cup \{h_i\}^8_{i=5}$ < 0 10 21 43 > $\cup$ {h_9, h_{10}, h_{11}, h_{11}}, < 0 1 4 9 > $\cup$ {h_1, h_2} < 0 6 25 41 > $\cup$ {h_3, h_4}, < 0 7 29 42 > $\cup$ {h_5, h_6} < 0 10 31 57 > $\cup$ {h_7, h_8} < 0 15 32 49 > $\cup$ {h_9, h_{10}}.

Lemma 4.5 There exists a $(v,5,2)$ packing design with a hole of size 4 for $v = 34, 54, 74, 94$.

Proof For a $(34,5,2)$ packing design with a hole of size 4 see [5, p.51]. For a $(74,5,2)$ packing design with a hole of size 4 take a $T[5,2,14]$ [18, p.278] and add four new points to the groups and on each group construct an $(18,5,4)$ packing design with a hole of size 4 [5, p.49]. For a $(54,5,2)$ and a $(94,5,2)$ packing design with a hole of size 4 take a $T[6,1,m]$ where $m = 5, 9$ respectively, theorem 2.1. Delete all but one point of the last group and inflate the design by a factor of two. Replace the blocks of this design by the blocks of $GD[5,2,2,10]$ and $GD[5,2,2,12]$ [18, p.284]. Finally add two new points to the groups and on the first five groups construct a $(12,5,2)$ and $(20,5,2)$ packing design with a hole of size 2 [5, p.49] and take these two points with the last group to be the hole of size 4.

Lemma 4.6 Let $v \equiv 15 \pmod{20}$ be a positive integer. Then $\sigma(v,5,3) = \psi(v,5,3)$.

Proof For $v = 15$ let $X = Z_{15}$ then the required blocks are

$\langle 0\ 1\ 3\ 7\ 10 \rangle \pmod{15}$ $\langle 0\ 1\ 2\ 5\ 7 \rangle \pmod{15}$

For $v = 35, 55, 75, 95$ let $X = Z_{v-7} \cup \{\infty_1, \infty_2\} \cup \{h_1, \ldots, h_7\}$ then the construction is as follows

1) On $Z_{v-7} \cup \{\infty_1, \infty_2\} \cup \{h_1, \ldots, h_4\}$ construct a $(v-1,5,2)$ packing design with a hole of size 4, say, $\{h_1, \ldots, h_4\}$ and assume that the pair $\{\infty_1, \infty_2\}$ appears at most once, lemma 4.5.

2) On $Z_{v-7} \cup \{\infty_1, \infty_2\} \cup \{h_1, \ldots, h_7\}$ construct a $(v+2,5,1)$ packing design with a hole of size 9 [17] where the hole is $\{\infty_1, \infty_2\} \cup \{h_1, \ldots, h_7\}$. In this design replace the points h_6 and h_7 by h_5.

3) To the blocks obtained in (1) and (2) add the blocks $\langle h_1\ h_2\ h_3\ h_4\ h_5 \rangle$ twice, $\langle \infty_1\ \infty_2\ h_1\ h_2\ h_5 \rangle$, $\langle \infty_1\ \infty_2\ h_3\ h_4\ h_5 \rangle$.

For $v \geq 115$, $v \neq 135$, write $v = 20m+4u+h+s$ where m, u, h and s are chosen as in lemma 4.4 with the difference that $4u+h+s = 15, 35, 55, 75, 95$. Now apply theorem 2.4 with $\lambda = 3$ to get the result.

For $v = 135$ apply theorem 2.2 with $n = 7$, $h = 3$ and $u = 3$.

Lemma 4.7 Let $v \equiv 19 \pmod{20}$ be a positive integer. Then $\sigma(v,5,3) = \psi(v,5,3)$.

Proof For $v = 19$ let $X = \{1, \ldots, 19\}$ then the blocks are

< 1 2 4 5 14 >, < 3 5 6 13 16 >, < 2 4 15 17 19 >, < 1 2 5 8 15 >

< 3 5 6 10 11 >, < 5 9 10 14 18 >, < 1 3 4 8 18 >, < 3 5 10 16 19 >

< 2 5 11 17 18 >, < 1 3 13 14 15 >, < 3 7 9 14 15 >, < 6 7 14 18 19 >

< 1 4 6 8 9 >, < 3 8 15 17 18 >, < 2 6 8 10 19 >, < 1 5 9 12 19 >

< 3 9 12 17 19 >, < 6 12 13 15 19 >, < 1 6 11 14 17 >, < 4 5 12 13 17 >

< 2 6 9 16 18 >, < 1 6 12 16 17 >, < 4 8 9 11 12 >, < 7 8 9 13 16 >

< 1 7 9 10 17 >, < 4 9 10 15 16 >, < 2 7 10 11 12 >, < 1 7 11 18 19 >

< 4 10 13 14 17 >, < 8 10 11 13 19 >, < 1 7 11 15 16 >, < 4 11 15 16 19 >

< 2 9 11 13 14 >, < 1 12 13 16 18 >, < 4 14 16 18 19 >, < 8 10 12 14 16 >

< 2 3 7 16 17 >, < 5 6 9 11 15 >, < 2 10 12 15 18 >, < 2 3 8 12 14 >

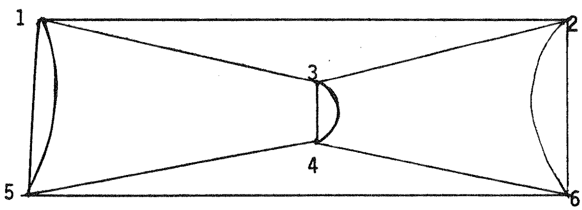
< 5 7 8 13 18 >, < 8 11 14 16 17 >, < 2 3 9 13 19 >, < 5 7 8 17 19 >

< 3 4 6 7 10 >, < 2 4 6 7 13 >, < 5 7 12 14 15 >, < 10 13 15 17 18 >

< 3 4 11 12 18>.

For all other values of v, $v \neq 239$, the construction is as follows.

1) Take two copies of a $(v-2,5,1)$ packing design with a hole of size 9, [17], and on the hole construct a $(9,5,2)$ packing design with $\psi(9,5,2)-1$ blocks [6]. Close observation of this design shows that the complement graph of this design consists of the following graph



2) Take a $(v+4,5,1)$ optimal packing design, $v+4 \neq 243$, [20]. Again, close observation of these designs show that the complement graph of these designs contains a subgraph on $n \geq 23$ vertices which is one cycle. So we may assume that the pairs $\{1,4\}$, $\{2,4\}$, $\{2,5\}$, $\{3,5\}$, $\{v-1, v+1\}$ and $\{v-1, v+2\}$ appear in zero blocks. Furthermore, assume we have the block $< v\ v+1\ v+2\ v+3\ v+4 >$. Delete this block and in all other blocks change $v+4$, $v+3$ to v and $v+2$, $v+1$ to $v-1$.

3) To the blocks obtained in (1) and (2) add the block < 1 2 3 4 6 >.

For $v = 239$ apply theorem 2.4 with $m = 11$, $u = 4$, $h = 3$, $s = 0$ and $\lambda = 3$.

Conclusion In this section we have shown that for all positive integers $v \equiv 3 \pmod 4$ $v \geq 7$ we have $\sigma(v,5,3) = \psi(v,5,3)$.

5. Packing Designs With Index 5

5.1 Packing of Order $v \equiv 3 \pmod 4$

Theorem 5.1 For all $v \equiv 3 \pmod{20}$, $v \neq 3$, we have $\sigma(v,5,5) = \psi(v,5,5)$. Furthermore there exists a (23,5,5) packing design with a hole of size 3.

Proof A $(v,5,5)$ packing design with $\psi(v,5,5)$ blocks may be constructed as follows.

1) Take a $B[v+2,5,1]$, lemma 2.1, and assume we have the following two blocks

   $< 1\ \ 2\ \ 3\ \ v\ \ v+2 >$, $< 4\ \ 5\ \ 6\ \ v-1\ \ v+1 >$

   In the first block change $v+2$ to 7 and in the second block change $v+1$ to 8 where 1,2, ...,7,8 are all arbitrary numbers. In all other blocks change $v+2$ to v and $v+1$ to $v-1$.

2) Take a $B[v-2,5,1]$, lemma 2.1, $v-2 \neq 21$, and assume we have the following two blocks

   $< 1\ \ 2\ \ 3\ \ 9\ \ 7 >$, $< 4\ \ 5\ \ 6\ \ 10\ \ 8 >$

   In the first block change 7 to v and in the second block change 8 to $v-1$.

   The above two steps give us a sort of a design such that {7,9} and {8,10} each appears exactly once; $\{7,v\}$ $\{8,v-1\}$ $\{9,v\}$ $\{10,v-1\}$ each appears exactly 3 times; $\{v-1,v\}$ appears exactly four times, and each other pair appears exactly twice.

3) Take a $(v,5,2)$ optimal packing with a hole of size 3, [5], such that the hole is $\{v-2, v-1, v\}$

4) Take a $(v,5,1)$ optimal packing, [20], which exists for all $v \equiv 3 \pmod{20}$, $v \neq 243$. The complement graph of this design contains a subgraph that is the circuit graph C_n where $n \geq 23$, we may assume that $\{7,v\}$, $\{8,v-1\}$ $\{9,v\}$ and $\{10,v-1\}$ are missing from the $(v,5,1)$ optimal packing design.

It is readily checked that the above four steps yield the blocks of a $(v,5,5)$ optimal packing design for all positive integers $v \equiv 3 \pmod{20}$ $v \neq 23$, 243.

For $v = 23$ the construction is as follows:

1) take a (23,5,2) minimal covering design [19]. In this design each pair appears in precisely two blocks except one pair, say, {22,23} that appears in 6 blocks.

2) take a (23,5,2) optimal packing design with a hole of size 3, say, {5,22,23} [6].

3) take a (23,5,1) optimal packing desing. The complement graph of this design is the circuit graph C_{23}, [20], so we may assume that the pairs {22,23} and {4,23} appear in zero blocks.

The above three steps give a design such that {22,23} appears in six blocks and each other pair in at most 5 blocks. To reduce this to five, assume in the (23,5,2) minimal covering design we have the block < 1 2 3 22 23 >. In this block change 23 to 5. Furthermore, assume in the (23,5,2) optimal packing design we have the block <1 2 3 4 5 >. In this block change 5 to 23.

Now it is easy to check that the above construction yields a (23,5,5) optimal packing design.

For $v = 243$ apply theorem 2.4 with $m = 11$, $\lambda = 5$, $h = 3$, $s = 0$ and $u = 5$.

For a (23,5,5) packing design with a hole of size 3 let $X = Z_{20} \cup H_3$. Then the required blocks are:

On $Z_{20} \cup \{h_1\}$ construct a B[21,5,1], lemma 2.1, and take the following blocks:

$< 0\ 4\ 8\ 12\ 16 > + i$, $i \in Z_4$, 3 times $< 0\ 3\ 10\ 13\ h_1 >$ half orbit

$< 0\ 1\ 2\ 3\ 5 >$ (mod 20), $< 0\ 1\ 7\ 12\ h_2 >$ (mod 20), $< 0\ 2\ 7\ 13\ h_3 >$ (mod 20)

$< \kappa\ \ \kappa+3\ \ \kappa+9\ \ \kappa+14\ \ f(\kappa) >$ $\kappa = 0,\ldots, 19$ where $f(\kappa) = h_1$ if $\kappa \equiv 0$ or 1 (mod 4), $f(\kappa) = h_2$ if $\kappa \equiv 2$ (mod 4) and $f(\kappa) = h_3$ if $\kappa \equiv 3$ (mod 4).

In the following lemma we give direct constructions for small values of v.

<u>Lemma 5.1</u> $\sigma(v,5,5) = \psi(v,5,5)$ for v=7, 27, 47, 67, 87.

<u>Proof</u> For $v = 7$, 47, 67, 87 the constructions are given in the following table.

For $v = 27$ the construction is as follows:

1) take a B[26,5,4], lemma 2.1;

2) take a (31,5,1) optimal packing design ([20], lemma 3.6 with $s = 8$). Assume in this design we have the block < 27 28 29 30 31 >. Delete this block and in all other blocks change 28, 29, 30 and 31 to 27.

v	Point Set	Base Blocks
7	$Z_2 \times Z_3 \cup H_1$	< (0,0)(0,1)(1,0)(1,2) h_1 >, < (0,0)(0,1)(1,0)(1,1) h_1 > < (0,0)(0,1)(1,0)(1,1)(1,2) >
47	$Z_{40} \cup H_7$	On $Z_{40} \cup H_5$ construct a B[45,5,1], lemma 2.1. Assume <h_1, ..., h_5> are in one block. Delete this block and take the following blocks <0 8 16 24 32> + i, i $\in Z_8$, <0 13 20 33> $\cup$ {h_6, h_7}, half orbit <0 1 2 4 10>, <0 3 10 24 28>, <0 5 14 20 27>, <0 5 17 28 h_1> <0 1 6 15 h_2>, <0 2 19 30 h_3>, <0 3 4 22 h_4>, <0 3 10 14 h_5> < 0 2 8 17 h_6 >, < 0 5 13 24 h_7 >.
67	$Z_{60} \cup H_7$	On $Z_{60} \cup H_5$ construct a B[65,5,1], lemma 2.1. Assume <h_1, ..., h_5> are in one block. Delete this block and take the following blocks <0 12 24 36 48> + i, i $\in Z_{12}$ <0 21 30 51> $\cup$ {h_6, h_7}, half orbit <0 1 3 5 11>, <0 7 14 26 42>, <0 1 3 7 23>, <0 5 14 27 45> <0 6 17 32 42>, <0 1 3 7 15>, <0 10 20 31 44>, <0 9 22 45 h_1> <0 8 25 41 h_2>, <0 8 27 39 h_3>, <0 17 20 46 h_4>, <0 1 5 28 h_5> < 0 5 18 43 h_6 >, < 0 9 28 39 h_7 >.
87	$Z_{80} \cup H_7$	On $Z_{80} \cup H_5$ construct a B[85,5,1], lemma 2.1. Assume <h_1, ..., h_5> are in one block. Delete this block. On Z_{80} construct an (80,5,1) covering, [21]. In this design each pair appears exactly once except the pairs {i, i+40}; i $\in Z_{40}$, each appears exactly twice. Take the following blocks <0 16 32 48 64> + i, i $\in Z_{16}$, <0 11 40 51> $\cup$ {h_6, h_7}, half orbit <0 5 28 38 50>, <0 1 3 7 17>, <0 11 26 50 62>, <0 1 3 7 21> <0 1 3 7 25>, <0 5 14 22 53>, <0 10 30 43 59>, <0 8 27 42 h_1> <0 9 34 53 h_2>, <0 13 37 54 h_3>, <0 5 28 37 h_4>, <0 12 25 45 h_5> < 0 8 31 52 h_6 >, < 0 11 26 45 h_7 >.

<u>Theorem 5.2</u> Let $v \equiv 7 \pmod{20}$ be a positive integer. Then $\sigma(v,5,5) = \psi(v,5,5)$.

<u>Proof</u> For $7 \le v \le 87$, the result follows from lemma 5.1

For $v \ge 107$, $v \ne 127$, simple calculations show that v can be written in the form $v = 20m + 4u + h + s$ where m, u, h and s are chosen so that the following 4 conditions hold

1) there exists a RMGD[5,1,5,5m], theorem 2.3,
2) $4u + h + s \equiv 7 \pmod{20}$ and $7 \le 4u + h + s \le 87$,
3) $0 \le u \le m-1$, $s \equiv 0 \pmod 4$ and $h = 3$
4) there exists a GD[5,5,{4,s*},4m+s], theorem 2.5,

Now apply theorem 2.4 with $\lambda = 5$ and the result follows.

For $v = 127$, apply theorem 2.2 with $u = 1$, $h = 3$ and $n = 7$.

<u>Theorem 5.3</u> Let $v \equiv 11$ or $15 \pmod{20}$ be a positive integer. Then $\sigma(v,5,5) = \psi(v,5,5)$.

Proof A $(v,5,5)$ packing design with precisely $\psi(v,5,5)$ blocks for all $v \equiv 11$ or 15 (mod 20) can be constructed by simply taking the blocks of a $B[v,5,2]$ and a $(v,5,3)$ optimal packing designs, lemma 2.1 and lemmas 4.4 and 4.6 respectively. Since a $B[15,5,2]$ does not exist, lemma 2.1, we need to construct a $(15,5,5)$ optimal packing design.

For this purpose let $X = Z_{15}$ then the required blocks are

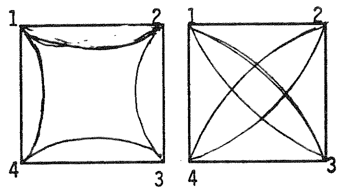
$< 0\ 3\ 6\ 9\ 12 > + i,\ i \in Z_3$ twice

$< 0\ 1\ 2\ 3\ 7 >$ (mod 15), $< 0\ 1\ 2\ 5\ 10 >$ (mod 15), $< 0\ 2\ 4\ 7\ 11 >$ (mod 15).

Lemma 5.2 Let $v \equiv 19$ (mod 20) be a positive integer and assume the following conditions are satisfied

1) $\sigma(v+4,5,1) = \psi(v+4,5,1)$ 2) $\alpha(v-1,5,4) = \phi(v-1,5,4)$

3) the excess graph $E(V,\beta)$ of the $(v-1,5,4)$ covering design consists of $v-4$ isolated vertices and one of the following graphs on the remaining 4 vertices, say, $\{1,2,3,4\}$.



Then $\sigma(v,5,5) = \psi(v,5,5)$.

Proof If the excess graph of the $(v-1,5,4)$ minimal covering design consists of $v-4$ isolated vertices and the graph on the bottom on the remaining four vertices, then a $(v,5,5)$ optimal packing design can be constructed as follows:

1) take the blocks of a $(v-1,5,4)$ minimal covering design and assume we have the block $< 1\ 2\ 3\ 4\ a >$ where a is an arbitrary number different from $\{1,2,3,4\}$. Delete this block.

2) take a $(v+4,5,1)$ optimal packing design. The complement graph of this design contains a circuit graph C_n where $n \geq 23$ [20], so we may assume that the pairs $\{1,3\}$ and $\{2,4\}$ are missing from this design. Furthermore, assume we have the block $< v\ \ v+1\ \ v+2\ \ v+3\ \ v+4 >$. Delete this block and in all the remaining blocks of the $(v+4,5,1)$ optimal packing design change $v+1$, $v+2$, $v+3$, and $v+4$ to v.

If the excess graph of the $(v-1,5,4)$ minimal covering design consists of $v-4$ isolated vertices and the top graph of the two graphs, on the remaining four vertices, then a $(v,5,5)$ optimal packing design can be constructed as follows

1) take a $(v-1,5,4)$ minimal covering design. Assume in this design we have the block $< 1\ 2\ 3\ 4\ 5 >$ where 5 is an arbitrary number. Delete this block.

Furthermore, assume in this design we have the block $< 6\ 7\ 8\ 1\ 4 >$ where $\{6,7,8\}$ are arbitrary numbers. In this block change 4 to 5.

2) take a $(\upsilon+4,5,1)$ optimal packing design. The complement graph of this design contains a circuit graph C_n where $n \geq 23$ [20], so we may assume that the pairs $\{1,2\}$, $\{2,3\}$ $\{3,4\}$ and $\{4,9\}$ appear in zero blocks. Assume in this design we have the block $< 6\ 7\ 8\ 9\ 5 >$. In this block change 5 to 4. Furthermore , assume in this design we have the block $< \upsilon\ \ \upsilon+1\ \ \upsilon+2\ \ \upsilon+3\ \ \upsilon+4 >$. Delete this block and in all other blocks change $\upsilon+1$, $\upsilon+2$, $\upsilon+3$ and $\upsilon+4$ to υ.

<u>Theorem 5.4</u> Let $\upsilon \equiv 19 \pmod{20}$ be a positive integer. Then $\sigma(\upsilon,5,5) = \psi(\upsilon,5,5)$.

<u>Proof</u> In [10] we have shown that for all $\upsilon-1 \equiv 18 \pmod{20}$ $\upsilon \not\equiv 98 \pmod{100}$, $\upsilon \neq 78$ there exists a $(\upsilon-1,5,4)$ covering design with a hole of size 8, 13 or 18. But for $n = 8$, 13, 18 there exists a $(n,5,4)$ minimal covering design such that their excess graphs is one of graphs described in lemma 5.2. We now show that for the other values there exists a $(\upsilon-1,5,4)$ covering design with a hole of size 8, 13, or 18.

For $\upsilon = 78$ see [4].

For $\upsilon \equiv 98 \pmod{100}$ take a $T[6,1,m]$ where $m \equiv 17 \pmod{20}$, theorem 2.1. Delete all but 11 points from last group and replace the blocks of the resultant design by the blocks of a $B[6,5,4]$ and $B[5,5,4]$, lemma 2.1. Add two points to the groups and on the first five groups construct a $(m+2,5,4)$ packing design with a hole of size 2 [12]. Finally, take these two points with the last group to be the hole of size 13. Now it is clear that for all $\upsilon-1 \equiv 18 \pmod{20}$ the excess graph of the $(\upsilon-1,5,4)$ minimal covering design is one of the graphs described in lemma 5.2.

On the other side a $(\upsilon+4,5,1)$ optimal packing design exists for all $\upsilon+4 \equiv 3 \pmod{20}$, $\upsilon+4 \neq 243$, [20]. Now apply lemma 5.2 to get the result for all $\upsilon \equiv 19 \pmod{20}$ $\upsilon \neq 239$.

For a $(239,5,5)$ optimal packing design apply theorem 2.4 with $\lambda=5$, $m=11$, $s=0$, $u=4$ and $h=3$.

5.2 Packing of order $\upsilon \equiv 2 \pmod{4}$

We start this section with the following simple but important observation

Lemma 5.3 (a) If there exists

1) a $(v,5,\lambda)$ covering design with $\phi(v,5,\lambda)$ blocks;

2) a $(v,5,\lambda')$ packing design with $\psi(v,5,\lambda')$ blocks;

3) $\phi(v,5,\lambda) + \psi(v,5,\lambda') = \psi(v,5,\lambda+\lambda')$;

4) the excess graph $E(V,\beta)$ of the covering design is isomorphic to a subgraph G of the complement graph, $C(V,\beta)$, of the packing design.

Then there exists a $(v,5,\lambda+\lambda')$ packing design with $\psi(v,5,\lambda+\lambda')$ blocks

(b) Similarly if there exists

1) a $(v,5,\lambda)$ covering design with a hole of size h;

2) a $(v,5,\lambda')$ packing design with a hole of size h;

3) the total number of blocks in these two designs is $\psi(v,5,\lambda+\lambda') - \psi(h,5,\lambda+\lambda')$;

4) the excess graph, $E(V\backslash H,\beta)$, of the covering design with a hole of size h is isomorphic to a subgraph G of the complement graph, $C(V\backslash H,\beta)$, of the packing design with a hole of size h.

Then there exists a $(v,5,\lambda+\lambda')$ packing design with a hole of size h.

Lemma 5.4 $\sigma(v,5,5) = \psi(v,5,5)$ for v = 22, 42, 62, 82. Furthermore, these packing designs have a hole of size 2.

Proof For $v = 22$ let $X = Z_{20} \cup \{a,b\}$ then the required blocks are

$<0\ 4\ 8\ 12\ 16> + i$, $i \in Z_4$, $<0\ 3\ 10\ 13> \cup \{a,b\}$ half orbit

$<0\ 1\ 2\ 3\ 5>$ (mod 20), $<0\ 1\ 6\ 8\ 13>$ (mod 20), $<0\ 2\ 8\ 11\ 14>$ (mod 20),

$<0\ 4\ 9\ 13\ a>$ (mod 20), $<0\ 1\ 5\ 11\ b>$ (mod 20).

For v = 42, 62, 82 the construction is as follows

1) Take a $B[v-1,5,2]$, lemma 2.1.

2) Take a $(v+1,5,2)$ optimal packing design [6]. It has a hole of size 3, say $\{v-1, v, v+1\}$. Now in all the blocks of the $(v+1,5,2)$ optimal packing design change $v+1$ to v.

3) Take a $(v,5,1)$ optimal packing design, v = 42, 62, 82, [9].

It is clear that the above three steps yield a $(v,5,5)$ optimal packing design for v = 42, 62, 82.

<u>Theorem 5.5</u> $\sigma(v,5,5) = \psi(v,5,5)$ for all positive integer $v \equiv 2 \pmod{20}$, $v \geq 22$.

<u>Proof</u> For v = 22, 42, 62, 82 the result follows from lemma 5.4. For $v \geq 102$ simple calculations show that v can be written in the form $v = 20m + 4u + h + s$ where m, u, h and s are chosen so that

1) there exists a RMGD[5,1,5,5m], theorem 2.3;
2) $4u + h + s \equiv 2 \pmod{20}$ and $22 \leq 4u + h + s \leq 82$;
3) $0 \leq u \leq m-1$, $s \equiv 0 \pmod 4$ and $h = 2$;
4) there exists a GD[5,5,{4,s*},4m+s], theorem 2.5.

Now apply theorem 2.4 with $\lambda = 5$ and the result follows.

<u>Lemma 5.5</u> $\sigma(v,5,5) = \psi(v,5,5)$ for v = 6, 26, 46, 66, 86.

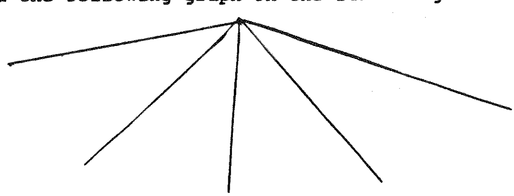
<u>Proof</u> For v = 6 take a B[6,5,4], lemma 2.1, with an optimal (6,5,1) packing, which has one block.

For v = 26 let $X = Z_{20} \cup H_6$. On $Z_{20} \cup H_5$ construct a B[25,5,1], lemma 2.1, such that < h_1 h_2 h_3 h_4 h_5 > is a block, which we delete. Furthermore, take the following base blocks under the action of the group Z_{20}:

<0 5 10 15 h_6> orbit length 5. < 0 1 2 3 h_1>, <0 1 3 8 h_2>, <0 2 7 13 h_3>, <0 3 9 12 h_4>, <0 4 8 13 h_5>, <0 4 8 14 h_6>.

For v = 46, 66, 86 a $(v,5,5)$ optimal packing design may be constructed as follows:

1. take a $(v,5,3)$ minimal covering design, [9]. Careful inspections show that the excess graph $E(V,\beta)$ of this covering design consists of a 1- factor on $v-6$ vertices and the following graph on the remaining 6 vertices



2. take a $(v,5,2)$ optimal packing design such that its complement graph $C(V,\beta)$ contains a subgraph G that is isomorphic to $E(V,\beta)$, the excess graph of $(v,5,3)$ minimal covering design, lemma 4.2. Now apply lemma 5.3 and the result follows.

<u>Theorem 5.6</u> $\sigma(v,5,5) = \psi(v,5,5)$ for all positive integers $v \equiv 6 \pmod{20}$

<u>Proof</u> For $6 \le v \le 86$ the result follows from lemma 5.5. For $v \ge 106$ the proof of this theorem is the same as theorem 5.2 with the difference that $4u + h + s \equiv 6 \pmod{20}$, $h = 6$, and $6 \le 4u + h + s \le 86$.

<u>Lemma 5.6</u> Let m, u and h ≥ 0 be positive even integers. If there exists (1) a GD[5,2,{m,u*}, 5m+u] (2) a (u+h,5,2) optimal packing design with $\frac{2(u+h)^2 - 2(u+h) + c(u+h) + d}{20}$ blocks where c and d are integers determined by u and h (3) a (m+h,5,2) packing design with a hole of size h with total number of blocks equal $\frac{2m^2 + 4hm + cm - 2m}{20}$. Then $\sigma(5m+u+h,5,2) = \psi(5m+u+h,5,2)$

<u>Proof</u> We need to show that the total number of blocks obtained by this construction is equal to $\psi(5m+u+h,5,2)$. But a GD[5,2,{m,u*}, 5m+u] has the following number of blocks $2(m(m-u) + \frac{3}{2}mu)$ (I)

A (u+h,5,2) optimal packing design has the following number of blocks

$$\frac{2(u+h)^2 - 2(u+h) + c(u+h) + d}{20} \qquad \text{(II)}$$

where c and d are integers deterimed by u and h, and a (m+h,5,2) packing design with a hole of size h has the following number of blocks (we are assuming that this number is an integer)

$$\frac{2m^2 + 4mh + cm - 2m}{20} \qquad \text{(III)}$$

where c is as above.

On the other hand, $\psi(5m+u+h,5,2) = \dfrac{2(5m+u+h)^2 - 2(5m+u+h) + c(5m+u+h) + d}{20}$ (IV)

where c and d are the same integers as in (II) since 5m+u+h and u+h are the same congruency modulo 10.

Now it is easily checked that the total number of blocks in (I), (II) and 5 times the number of blocks in (III) is equal to the total number of blocks in (IV).

Lemma 5.7 Let $v \equiv 10$ or $14 \pmod{20}$, $v \neq 34$ be a positive integer less than 100. Then there exists a $(v,5,2)$ optimal packing design such that the complement graph of these designs contains a subgraph that is a 1-factor.

Proof For v = 10, 14, 30, 90 see [5, p.51].

For v = 70 let X = $Z_{68} \cup \{a,b\}$, then take the following base blocks under the action of the group Z_{68}.

<0 1 3 8 22>, <0 4 17 35 44>, <0 10 25 36 48>, <0 1 3 7 18>, <0 5 24 30 40>
<0 9 22 36 48>, <0 8 29 45> ∪ {a,b}.

For v = 50, 54, 74 and 94 take a GD[5,2,{m,u*}, 5m+u] where m, u and h are choosen as prescribed in the table below (see lemma 2.1 of [5, p. 46] for the existence of a GD[5,2, {m,u*}, 5m+u]). Adjoin a set H of h points to the groups and on the first five groups construct a (m+h,5,2) packing design with a hole of size h [5, p. 48] and take these h points with the last group as a block which we delete since the total number of points is less than five. Now apply lemma 5.6 to get the result.

v	m	u	h	Lemma	v	m	u	h	Lemma
50	10	0	0	5.6	74	14	0	4	5.6
54	10	2	2	5.6	94	18	2	2	5.6

Note that our constructions are correct provided that: the (10,5,2) optimal packing design; the (12,5,2) packing design with a hole of size 2; the (18,5,2) packing design with a hole of size 4, and the (20,5,2) packing design with a hole of size 2, their complement graph has a complement subgraph that is

1-factor. This can easily be checked. For the (18,5,2) packing design with a hole of size 4, the 1-factor on {5,...,18} is {{5,17} {6,12} {7,9} {8,11} {10,16} {13,18} {14,15}}.

Lemma 5.8 $\sigma(v,5,5) = \psi(v,5,5)$ for all $v \equiv 10$ or $14 \pmod{20}$ and $10 \le v \le 94$, $v \neq 34$.

Proof A $(v,5,5)$ optimal packing design for $v \equiv 10$ or $14 \pmod{20}$ and $v \le 94$ can be constructed as follows.

1) take a $(v,5,3)$ minimal covering design [9]. The excess graph, $E(V,\beta)$, of each of these designs is a 1- factor.

2) take a $(v,5,2)$ optimal packing design such that the compliment graph of these designs contains a subgraph which is 1-factor (lemma 5.7). Since $\alpha(v,5,3) = \phi(v,5,3)$ and $\sigma(v,5,2) = \psi(v,5,2)$ for such v; and $\alpha(v,5,3) + \psi(v,5,2) = \psi(v,5,5)$ it follows that $\sigma(v,5,5) = \psi(v,5,5)$.

Theorem 5.7 $\sigma(v,5,5) = \psi(v,5,5)$ for all positive integers $v \equiv 10$ or $14 \pmod{20}$ with the possible exception of $v = 34$.

Proof For $14 \le v \le 94$, $v \equiv 10$ or $14 \pmod{20}$ the result follows from lemma 5.8. For $v \ge 110$, $v \neq 130, 134, 214$, the proof of the theorem is the same as theorem 5.5 with the difference that $4u + h + s = 10, 30, 50, 70, 90$ if $v \equiv 10 \pmod{20}$ and $4u + h + s = 14, 54, 74, 94$ if $v \equiv 14 \pmod{20}$. For $v = 130, 134$ apply theorem 2.2 with $h = 2$, $n = 7$ and $u = 2$ and 3 respectively.

For $v = 214$ take a T[6,5,10], [18, p.278], and delete 7 points from the last group. Inflate this design by a factor of 4, that is, replace each block of size 5 and 6 by the blocks of a GD[5,1,4,20] and GD[5,1,4,24] respectively, lemma 2.1. Add two points to the groups and on the first 5 groups construct a (42,5,5) packing design with a hole of size 2 (This design exists by lemma 5.4); and on the last group construct a (14,5,5) optimal packing design.

Lemma 5.9 $\sigma(v,5,5) = \psi(v,5,5)$ for $v = 18, 38, 58, 78, 98$.

Proof For υ = 18 let X = {1,2,...,18} then the required blocks are

< 1 2 3 4 10 >,	< 4 5 13 15 16 >,	< 1 2 8 14 18 >,	< 4 5 15 16 18 >
< 1 2 8 14 15 >,	< 4 8 10 11 17 >,	< 1 2 8 12 15 >,	< 4 9 10 14 15 >
< 1 2 11 15 16 >,	< 4 9 11 12 14 >,	< 1 3 5 9 14 >,	< 4 10 13 14 18 >
< 1 3 5 6 7 >,	< 5 6 11 13 16 >,	< 1 3 10 13 18 >,	< 5 7 8 10 15 >
< 1 3 11 14 16 >,	< 5 8 12 17 18 >,	< 1 4 6 8 18 >,	< 5 9 10 11 18 >
< 1 4 7 16 18 >,	< 5 11 13 14 15 >,	< 1 4 7 12 13 >,	< 6 7 8 9 10 >
< 1 4 9 11 17 >,	< 6 7 8 11 14 >,	< 1 5 6 9 15 >,	< 6 7 10 13 18 >
< 1 5 7 11 16 >,	< 6 8 9 11 13 >,	< 1 5 8 12 17 >,	< 6 10 15 16 17 >
< 1 6 9 13 17 >,	< 6 13 14 16 18 >,	< 1 6 10 11 17 >,	< 7 8 9 14 16 >
< 1 7 12 13 17 >,	< 7 10 12 14 16 >,	< 1 9 10 12 15 >,	< 7 11 15 17 18 >
< 1 10 14 16 18 >,	< 8 9 12 13 16 >,	< 2 3 5 6 10 >,	< 8 10 13 15 17 >
< 2 3 8 10 11 >,	< 9 11 12 15 16 >,	< 2 3 9 13 16 >,	<10 12 13 14 17 >
< 2 3 9 13 17 >,	< 2 4 5 7 13 >,	< 2 4 6 12 14 >,	< 2 4 7 10 11 >
< 2 4 11 12 13 >,	< 2 5 8 13 14 >,	< 2 5 10 12 16 >,	< 2 5 11 17 18 >
< 2 6 7 9 18 >,	< 2 6 14 15 17 >,	< 2 7 9 16 17 >,	< 2 7 9 15 18 >
< 2 12 16 17 18 >,	< 3 4 6 12 15 >,	< 3 4 6 8 16 >,	< 3 4 7 15 17 >
< 3 4 8 16 17 >,	< 3 5 7 14 17 >,	< 3 5 9 10 12 >,	< 3 6 12 15 18 >
< 3 7 8 13 15 >,	< 3 7 11 12 14 >,	< 3 8 11 12 18 >,	< 3 9 14 17 18 >
< 3 11 13 15 18 >,	< 4 5 6 14 17 >,	< 4 5 8 9 18 >.	

For υ = 38, 58, 78 the construction is as follows

1) take a (υ-1,5,4) optimal packing design, [12];

2) take a (υ+4,5,1) optimal packing design, [9]. Assume we have the block < υ υ+1 υ+2 υ+3 υ+4 >. Delete this block and in all other blocks change the points υ+1, υ+2, υ+3, υ+4 to υ.

For υ = 98 let X = $Z_{80} \cup H_{18}$. Then the construction is as follows:

1) On $Z_{80} \cup H_9$ construct an (89,5,1) packing design with a hole of size 9,[17].

2) On $Z_{80} \cup \{h_i\}^{18}_{i=10}$ construct an (89,5,1) packing design with a hole of size 9.

3) Take the following base blocks under the action of the group Z_{80}

<0 2 11 30 59>, <0 1 4 14 h_1>, <0 5 12 37 h_2>, <0 6 29 53 h_3>, <0 8 34 52 h_4>
<0 15 31 50 h_5>, <0 17 38 58 h_6>, <0 1 3 7 h_7>, <0 5 13 23 h_8>, <0 9 35 47 h_9>
<0 11 27 55 h_{10}>, <0 14 31 51 h_{11}>, <0 15 34 56 h_{12}>, <0 1 3 7 h_{13}>, <0 5 13 30 h_{14}>
<0 9 21 48 h_{15}>, <0 10 36 47 h_{16}>, <0 14 34 49 h_{17}>, <0 16 38 56 h_{18}>.

Theorem 5.8 $\sigma(\upsilon,5,5) = \psi(\upsilon,5,5)$ for all positive integers $\upsilon \equiv 18 \pmod{20}$.

Proof For $18 \le \upsilon \le 98$ see lemma 5.9. For $\upsilon \ge 118$, $\upsilon \ne 138$ the proof of this theorem is the same as theorem 5.5 with the difference that
$4u + h + s \equiv 18 \pmod{20}$, $18 \le 4u + h + s \le 98$.

For υ = 138 apply theorem 2.2 with n = 7, h = 2 and u = 4.

5.3 Packing of order $\upsilon \equiv 0 \pmod 4$

Theorem 5.9 Let $\upsilon \equiv 16 \pmod{20}$ be a positive integer. Then $\sigma(\upsilon,5,5) = \psi(\upsilon,5,5)$.

Proof The blocks of a $(v,5,5)$ optimal packing design for all positive integers $v \equiv 16 \pmod{20}$, may be constructed as follows.

1) take a $B[v-1,5,4]$, lemma 2.1;

2) take a $(v+4,5,1)$ optimal packing design which is constructed by taking a $B[v+5,5,1]$ and deleting the point $v+5$ and all the blocks containing this point. Assume in the $(v+4,5,1)$ optimal packing design we have the block < v $v+1$ $v+2$ $v+3$ $v+4$ >. Delete this block and in all the remaining blocks change $v+1$, $v+2$, $v+3$ and $v+4$ to v.

Lemma 5.10 There exists a $(24,5,5)$ packing design with a hole of size 4.

Proof Let $X = Z_{20} \cup H_4$ then take the following base blocks under the action of the group Z_{20}

<0 4 8 12 16> orbit of length 4, three times <0 2 3 5 9>, <0 1 2 4 h_1>, <0 1 6 13 h_2>, <0 2 7 13 h_3>, <0 3 9 12 h_4>, <0 1 6 11> $\cup \{h_i\}^4_{i=1}$.

Theorem 5.10 Let $v \equiv 4 \pmod{20}$ be a positive integer greater than 4. Then $\sigma(v,5,5) = \psi(v,5,5)$.

Proof For $v = 24, 44, 64, 84$ the construction is as follows:

1) take a $(v-1,5,1)$ optimal packing design, [20].

2) take a $B[v+1,5,1]$, lemma 2.1. Assume we have the block <1 2 3 v $v+1$>. In this block change $v+1$ to 5, where $\{1,2,3,5\}$ are arbitrary numbers, and in all other blocks change $v+1$ to v.

3) take a $(v,5,3)$ optimal packing design [9] and assume that the pairs $\{4,v\}$ and $\{5,v\}$ each appears at most twice (close observation of these designs show that we may assume this). Furthermore, assume in this design we have the block <1 2 3 4 5>. In this block change 5 to v. Now it is easily checked that the above three steps yield a $(v,5,5)$ optimal packing design for $v = 24, 44, 64, 84$.

For $v \geq 124$, $v \neq 144, 224$ simple calculations show that v can be written in the form $v = 20m+4u+h+s$ where m, u, h and s are chosen as in theorem 5.5 with the difference that $4u+h+s = 24, 44, 64, 84$ and $h = 4$.

Now apply theorem 2.4 with $\lambda = 5$ and the result follows.

For $v = 104, 144, 224$ apply theorem 2.5 with $m = 5, 7, 11$ respectively.

Theorem 5.11 Let $v \equiv 0$, 8 or 12 (mod 20) be a positive integer greater than zero. Then $\sigma(v,5,5) = \psi(v,5,5)$ with the possible exception of $v = 28, 32$.

Proof We first prove the theorem for $8 \le v \le 100$, $v \ne 28, 32$. For $8 \le v \le 100$, $v \ne 20, 28, 32$ a $(v,5,5)$ optimal packing design can be constructed by taking the blocks of a $(v,5,3)$ and a $(v,5,2)$ optimal packing design [9], [5].

For $v = 20$ let $X = Z_{20}$ then the blocks are
<0 4 8 12 16> + i, $i \in Z_4$, 3 times <0 1 4 10 15> (mod 20), <0 2 7 10 13> (mod 20), <0 1 2 3 5> (mod 20), <0 1 7 9 14> (mod 20).

For $v \ge 100$ $v \ne 128, 132, 208, 212$, simple calculations show that v can be written in the form $v = 20m+4u+h+s$ where m, u, h and s are chosen as in theorem 5.10 with the difference $4u+h+s \equiv 0$, 8 or 12 (mod 20), $8 \le 4u+h+s \le 92$, $4u+h+s \ne 28, 32$. Now apply theorem 2.4 with $\lambda=5$ and the result follows.

For $v = 128, 132$ apply theorem 2.2 with $n = 7$, $h = 0$ and $u = 2, 3$ respectively.

For $v = 208, 212$ take a T[6,5,10], [18, p.278], and delete all but u points from last group where u = 2, 3, respectively. Inflate this design by a factor of 4, that is, replace all blocks of size 5 and 6 by the blocks of a GD[5,1,4,20] and GD[5,1,4,24] respectively, lemma 2.1. Finally on the groups construct a (n,5,5) optimal packing design where n = 40, 8, 12.

7. Conclusion

We have shown that $\sigma(v,5,5) = \psi(v,5,5)$ for all positive integers v, $v \ge 5$ with the possible exception of $v = 28, 32, 34$.

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