Latin and Semi-Latin Factorizations of Complete Graphs and Support Sizes of Quadruple Systems *

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Abstract

In this paper, we introduce the notions of Latin and semi-Latin factorizations of graphs and their support sizes. We essentially determine the set support sizes of Latin and semi-Latin factorizations of complete graphs. Utilizing these results we determine the set $QSS(8m, \lambda)$ of support sizes of quadruple systems of order $8m$ and index λ for $m \geq 6$ with at most 5 possible omissions for each $m \equiv 0 \pmod 3$.

1 Introduction

Let X be a finite set and k be a positive integer. We denote by $P_k(X)$ the set of all k-subsets of X. Suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are two collections of the elements of $P_k(X)$ and m is a positive integer. The collection of the elements of $\mathcal{B}_1$ and $\mathcal{B}_2$ will be denoted by $\mathcal{B}_1 + \mathcal{B}_2$ and m copies of $\mathcal{B}_1$ is denoted by $m\mathcal{B}_1$. The set of distinct elements of $\mathcal{B}_1$ is called the support of $\mathcal{B}_1$ and is denoted by $\mathcal{B}_1^*$. The number $b^* = |\mathcal{B}_1^*|$ is called the support size of $\mathcal{B}_1$. Let X_1 and X_2

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be two disjoint sets, and let k_1 and k_2 be two positive integers. Also, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two collections of the elements of $P_{k_1}(X_1)$ and $P_{k_2}(X_2)$, respectively. Then, we will adopt the following notation:

$$\mathcal{B}_1 * \mathcal{B}_2 = \{A \cup B | A \in \mathcal{B}_1 \ \& B \in \mathcal{B}_2\}.$$

Clearly $\mathcal{B}_1 * \mathcal{B}_2$ is a collection of the elements of $P_{k_1+k_2}(X_1 \cup X_2)$.

A quadruple system $QS(v, \lambda)$ of order v and index λ is an ordered pair $D = (X, \mathcal{B})$ in which X is a v-set and $\mathcal{B}$ is a collection of the elements of $P_4(X)$ (called quadruples or blocks) such that every $A \in P_3(X)$ appears in exactly λ (not necessarily distinct) blocks. A $QS(v, \lambda)$ with no repeated blocks is called simple. A $QS(v, 1)$ is called a Steiner quadruple system of order v and will be denoted by $SQS(v)$.

A λ-factor in a multigraph G is a submultigraph F which is spanning and λ-regular. A λ-factorization of a multigraph G is a partition of edges of G into λ-factors. For a simple graph G, the multigraph λG is obtained by repeating each edge λ times. Let G be a 1-factorable graph of degree d. An $m\lambda$-factorization $\Gamma = \{F_1, \ldots, F_d\}$ of λG is called completely decomposable if there exist λ one-factorizations of G, e.g. $\Gamma_i = \{F_1^i, \ldots, F_d^i\}$, $1 \leq i \leq \lambda$, such that for every $1 \leq j \leq d$, F_j is the union of F_j^i's $(1 \leq i \leq \lambda)$. The support size of a λ-factor is the number of distinct edges in the factor, and the support size of a λ-factorization is the sum of the support sizes of its factors. We denote by $CS(G, \lambda)$ the set of the support sizes of completely decomposable λ-factorizations of λG. For simplicity, $CS(K_n, \lambda)$, and $CS(K_{n,n}, \lambda)$ will be denoted by $CS(n, \lambda)$, and $C(n, \lambda)$, respectively. These sets have been completely determined in [1,3] and the main results are as follows.

- Given n and λ, $n \geq 5$,

$$C(n, \lambda) = \begin{cases} \{n^2, \ldots, \min\{n, \lambda\}.n^2\} \setminus A, & \text{if } \lambda \neq n, \\ \{n^2, \ldots, \min\{n, \lambda\}.n^2\} \setminus B, & \text{otherwise,} \end{cases}$$

  where $A = \{n^2 + i | i = 1, 2, 3, 5\}$ and $B = \{n^2 + i, n^3 - i | i = 1, 2, 3, 5\}$.

- For given n and λ, $n \geq 4$,

$$CS(2n, \lambda) = \begin{cases} \{m, \ldots, M\} \setminus A, & \text{if } \lambda \neq 2n - 1, \\ \{m, \ldots, M\} \setminus B, & \text{otherwise,} \end{cases}$$

  where $m = n(2n - 1)$, $M = \min\{2n - 1, \lambda\}.m$, $A = \{m + i | i = 1, 2, 3, 5\}$, $B = \{m + i, M - i | i = 1, 2, 3, 5\}$.

Let G be a d-regular graph on $2n$ vertices which is 1-factorable, and let $\psi(G,\lambda)$ be the set of all λ-factors of λG. A Latin λ-factorization of λG is a $d \times d$ matrix $F = (F_{ij})$ with entries in $\psi(G,\lambda)$ such that for every $1 \leq i \leq d$, $\mathcal{F}_i = \{F_{ij} \mid 1 \leq j \leq d\}$ and $\Gamma_i = \{F_{ji} \mid 1 \leq j \leq d\}$ are two λ-factorizations of λG. The support size of F is the sum of the support sizes of its entries. A Latin λ-factorization of λG is called completely decomposable if there exists λ Latin 1-factorizations of G, e.g. $F^l = (F^l_{ij})$, $1 \leq l \leq \lambda$, such that $F_{ij} = \sum_{l=1}^{\lambda} F^l_{ij}$, for $1 \leq i,j \leq d$. The set of the support sizes of completely decomposable Latin λ-factorization of λG will be denoted by $LF(G,\lambda)$.

Let d be an odd integer and denote $d\Sigma 2 = \{\ \{i,j\} \mid 1 \leq i,j \leq d+1\}$. A semi-Latin λ-factorization of λG is a function $\mathcal{F} : d\Sigma 2 \rightarrow \psi(G,\lambda)$, $\{i,j\} \rightarrow F_{\{i,j\}}$ such that for every $1 \leq i \leq d$, $\{F_T \mid i \in T \in d\Sigma 2\}$ is a λ-factorization of λG. For simplicity, $F_{\{i,j\}}$ will be denoted by F_{ij}. The support size of F is the sum of the support sizes of F_{ij}'s. The set of the support sizes of semi-Latin λ-factorization of λG will be denoted by $SLF(G,\lambda)$. For the sake of simplicity, we denote $SLF(K_{2n},\lambda)$ by $SLF(2n,\lambda)$.

A (p,λ)-pattern is a $p \times p$ matrix with entries in nonnegative integers and with constant line sum λ. A $(p,1)$-pattern is called a permutation matrix. It is well known that every (p,λ)-pattern is a sum of λ(not necessarily distinct) permutation matrices. If $\{r_1,\ldots,r_n\}$ is any reordering of $\{1,\ldots,n\}$, then the permutation matrix $P = (\delta_{r_ij})$ will be denoted by $(r_1,\ldots,r_n)$. Support size of a (p,λ)-pattern is the number of its nonzero entries. Let $S_p(p,\lambda)$ denote the set of possible support sizes for (p,λ)-patterns. In [3] it is proved that if $p \geq 3$, then

$$S_p(p,\lambda) = \begin{cases} \{p,\ldots,\min\{\lambda p,p^2\}\} \setminus \{p+1\} & \text{if } \lambda \neq p, \\ \{p,\ldots,p^2\} \setminus \{p+1,p^2-1\} & \text{otherwise.} \end{cases}$$

In this paper we intend to determine the set $QSS(v,\lambda)$ of support sizes of quadruple systems of order v and index λ. First, we mention some well known results. Colbourn and Hartman [2], and Hartman and Yehudai[4] have completely determined the set $J(v)$ of possible intersections of two Steiner quadruple systems for $v \neq 14, 26$. In this way they have essentially determined the set $QSS(v,2)$ for $v \equiv 2$ or $4 \pmod 6$ with $v \neq 14, 26$. Also in [2], the set of possible intersections for a special class of 3-wise balanced designs is completely determined, and utilizing this result they obtained some partial result concerning $QSS(v,3)$ for $v \equiv 0 \pmod 6$. In this paper we essentially determine $LF(K_{2n},\lambda)$, $LF(K_{n,n},\lambda)$ and $SLF(K_{4n},\lambda)$ for $n \geq 12$ and then utilizing these results we essentially determine $QSS(v,\lambda)$ for $v \equiv 0 \pmod 8$ with $v > 48$.

2 Latin Factorizations

In this section, we develop some recursive methods to construct Latin factorizations with different support sizes, and then utilizing them we completely determine $LS(K_{2n}, \lambda)$ and $LF(K_{n,n}, \lambda)$ for $n \geq 12$.

2.1. Recursive Constructions

Throughout this section, we suppose that G is a simple m-regular graph on $2n$ vertices which is 1-factorable. Let G_1 be a subgraph of G such that both G_1 and $G_2 = G \setminus G_1$ are 1-factorable, and $deg(G_1) \leq m/2$. Denote $d = deg(G_1)$, and $k = deg(G_2)$, so $m = d + k$, and $d \leq k$. Our first task is to show that any Latin 1-factorization of G_1 can be embedded in a Latin 1-factorization of G. Let $\mathcal{H} = (H_{ij})$ be any Latin 1-factorization of G_1, and let $A = (a_{ij})$ be any Latin square of order m such that $1 \leq a_{ij} \leq d$, for $1 \leq i, j \leq d$ (since $2d \leq m$ this is possible). Since both of G_1 and G_2 are 1-factorable, we can form a 1-factorization $\mathcal{F} = \{F_1, \dots, F_m\}$ of G such that $\{F_1, \dots, F_d\}$ is a 1-factorization of G_1 and $\{F_{d+1}, \dots, F_m\}$ is a 1-factorization of G_2. Define

$$L_{ij} = \begin{cases} H_{ij}, & \text{if } 1 \leq i, j \leq d, \\ F_{a_{ij}}, & \text{otherwise.} \end{cases}$$

The proof of the following lemma is straightforward and so it is omitted.

Lemma 2.1 $\mathcal{L} = (L_{ij})$ is a Latin 1-factorization of G. □

Let σ be any permutation on $\{1, \dots, d\}$ and let η be any permutation on $\{d+1, \dots, m\}$. Define

$$K_{ij} = \begin{cases} L_{ij}, & \text{if } \ 1 \leq i \leq d & \text{and } \ 1 \leq j \leq d, \\ L_{\sigma(i)j}, & \text{if } \ 1 \leq i \leq d & \text{and } \ d+1 \leq j \leq m, \\ L_{\eta(i)j}, & \text{if } \ d+1 \leq i \leq m & \text{and } \ 1 \leq j \leq m. \end{cases}$$

The proof of the following lemma is straightforward.

Lemma 2.2 $\mathcal{K} = (K_{ij})$ is a Latin 1-factorization of G. □

Lemma 2.3 Let $\lambda \geq 2$. If $r \in CLF(G_1, \lambda)$, $s \in S_p(d, \lambda)$, $2 \leq p \leq \min\{d, \lambda\}$, and $q \in \{0, \dots, k\} \setminus \{1, k-1\}$, then

(i) $b = r + skn + (kp - q)mn \in CLF(G, \lambda)$.

(ii) $b + tnm^2 \in CLF(G, \lambda + t)$, for $t = 1, \ldots, k$.

Proof. Let $B = (b_{ij})$ be any $d \times k$ Latin rectangle whose first two rows are

$$\begin{array}{cccccccc} 1 & \ldots & q-1 & q & q+1 & \ldots & k-1 & k \\ 2 & \ldots & q & 1 & q+2 & \ldots & k & q+1 \end{array}$$

if $q \notin \{0, k\}$ and

$$\begin{array}{cccc} 1 & \ldots & k-1 & k \\ 2 & \ldots & k & 1 \end{array}$$

otherwise. Define λ permutation matrices $Q^1, \ldots, Q^\lambda$ by

$$Q^1 = \begin{cases} (1, \ldots, k-1, k), & \text{if } q = 0, \\ (2, \ldots, k, 1), & \text{if } q = k, \\ (2, \ldots, q, 1, q+1, \ldots, k-1, k), & \text{if } q \in \{2, \ldots, k-2\} \end{cases}$$

$$Q^l = \begin{cases} (b_{l1}, \ldots, b_{lk}), & \text{if } 2 \le l \le p, \\ (b_{p1}, \ldots, b_{pk}), & \text{if } p < l \le \lambda. \end{cases}$$

Then $Q = \sum_{i=1}^{\lambda} Q^i$ is a (k, λ)-pattern with support size $kp - q$. Let $C = (c_{ij})$ be any Latin square of order d and define a $d \times m$ matrix $D = (d_{ij})$ by

$$d_{ij} = \begin{cases} c_{ij}, & \text{for } 1 \le j \le d, \\ b_{i(j-d)} + d, & \text{for } d < j \le m, \end{cases} \qquad 1 \le i \le d.$$

Clearly D is a $d \times m$ Latin rectangle, and so it can be completed in a Latin square $A = (a_{ij})$ of order m. Let $P^1, \ldots, P^\lambda$ be permutation matrices of order d such that $P = \sum_{i=1}^{\lambda} P^i$ has exactly s nonzero entries. Let $\mathcal{M} = (M_{ij})$ be a completely decomposable Latin λ-factorization with support size r of λG_1. Since $\mathcal{M}$ is completely decomposable, we can find λ Latin 1-factorization of G_1, e.g. $\mathcal{M}_l = (M_{ij}^l)$, $l = 1, \ldots, \lambda$, such that $M_{ij} = \sum_{l=1}^{\lambda} M_{ij}^l$, for $1 \le i, j \le d$. Define L_{ij}'s as in Lemma 2.1. For every $1 \le l \le \lambda$, denote $P^l = (p_{ij}^l)$ and $Q^l = (q_{ij}^l)$. Also denote $P = (p_{ij})$ and $Q = (q_{ij})$. For every $1 \le l \le \lambda$, define:

$$N_{ij}^l = \begin{cases} M_{ij}^l & \text{if } \ 1 \le i \le d \ \text{ and } \ 1 \le j \le d, \\ \sum_{v=1}^{d} p_{iv}^l L_{vj}, & \text{if } \ 1 \le i \le d \ \text{ and } \ d+1 \le j \le d, \\ \sum_{v=1}^{k} q_{(i-d)v}^l L_{(v+d)j}, & \text{if } \ d+1 \le i \le m \ \text{ and } \ 1 \le j \le m. \end{cases}$$

By Lemma 2.2, for every $1 \leq l \leq \lambda$, $\mathcal{N}_l = (N^l_{ij})$ is a Latin 1-factorization of G. Thus if we put

$$N_{ij} = \sum_{l=1}^{\lambda} N^l_{ij}, \ 1 \leq i,j \leq m,$$

then $\mathcal{N} = (N_{ij})$ is a completely decomposable Latin λ-factorization of λG. To prove (i), we must show that the support size of $\mathcal{N}$ is b. For every $1 \leq i \leq k$, we denote by r_i the number of nonzero entries of ith row of Q, and for every $1 \leq i \leq d$, we denote by s_i the number of nonzero entries of ith row of P. Clearly, we have $kp - q = \sum_{i=1}^{k} r_i$, and $s = \sum_{i=1}^{d} s_i$. Now we compute the support size of each of N_{ij}'s.

If $1 \leq i \leq d$, and $d+1 \leq j \leq m$, then the support of N_{ij} consists of s_i edge-disjoint 1-factors of G, and consequently the support size of N_{ij} is equal to ns_i.

If $d+1 \leq i \leq m$, and $1 \leq j \leq m$, then the support of N_{ij} consist of r_{i-d} edge-disjoint 1-factors of G, and consequently the support size of N_{ij} is equal to nr_{i-d}.

The above considerations show that the support size of $\mathcal{N}$ is equal to

$$r + \sum_{i=1}^{d} \sum_{j=d+1}^{m} ns_i + \sum_{i=d+1}^{m} \sum_{j=1}^{m} t_{i-d}n = r + skn + (kp-q)mn = b.$$

This proves (i).

To prove (ii), define m permutations $\sigma_1, \ldots, \sigma_m$ by

$$\sigma_i(j) = a_{ij}, \quad \text{for } 1 \leq i,j \leq m.$$

It is clear that for every $1 \leq l \leq m$, $(L_{\sigma(l)j})$ is a Latin 1-factorization of G. Let

$$R_{ij} = N_{ij} + \sum_{l=d+1}^{d+t} L_{\sigma_l(i)j}, \quad \text{for } 1 \leq i,j \leq m.$$

Then $\mathcal{R} = (R_{ij})$ is a completely decomposable $(\lambda+t)$-factorization of $(\lambda+t)G$. To complete the proof of (ii), we have to show that the support size of $\mathcal{R}$ is equal to $b + tnm^2$, and to do this it suffices to show that for every i and j $(1 \leq i,j \leq m)$ N_{ij} and $N'_{ij} = \sum_{l=d+1}^{m} L_{\sigma_l(i)j}$ are edge-disjoint. We consider the followoing three cases:

Case (i) $1 \leq i,j \leq d$. In this case, we have $N'_{ij} = \sum_{l=d+1}^{m} L_{lj} = \sum_{l=d+1}^{m} F_l = G_2$. On the other hand, $N_{ij} = M_{ij}$ is a λ-factor of λG_1. Therefore N_{ij} and N_{ij}' are edge-disjoint.

Case (ii) $1 \leq i \leq d$ and $d+1 \leq j \leq m$. For every $1 \leq l \leq \lambda$, we have $N^l_{ij} = \sum_{v=1}^{d} P^l_{iv} L_{vj}$. Thus N^l_{ij} is a subgraph of $\sum_{v=1}^{d} L_{vj} = \sum_{l=1}^{d} L_{\sigma_l(i)j}$. Hence, for every $1 \leq l \leq \lambda$, N^l_{ij} and N_{ij}' are edge-disjoint. Consequently $N_{ij} = \sum_{l=1}^{\lambda} N^l_{ij}$ and N_{ij}' are edge-disjoint.

Case (iii) $d+1 \le i \le m$ and $1 \le j \le m$. For every $1 \le l \le \lambda$, we have $N^l_{ij} = \sum_{v=1}^{k} q^l_{(i-d)v} L_{(v+d)j}$. And by definition of A and Q^l's, we have

$$q^l_{(i-d)v} \neq 0 \iff v \in \{b_{l(i-d)} | 1 \le l \le d\} \iff v+d \in \{a_{li} | 1 \le l \le d\} = \{\sigma_l(i) | 1 \le l \le d\}.$$

Thus for every $1 \le l \le \lambda$, N^l_{ij} is a subgraph of $\sum_{l=1}^{d} L_{\sigma_l(i)j}$, and consequently N^l_{ij} and N_{ij}' are edge-disjoint. Therefore $N_{ij} = \sum_{l=1}^{\lambda} N^l_{ij}$ and N_{ij}' are edge-disjoint.

These observations show that the support size of $\mathcal{R}$ is equal to $b + nm^2$. □

Lemma 2.4 Let G_3 and G_4 be two simple 1-factorable graph of the same degree d on two disjoint sets of vertices. If $r \in CLF(G_3, \lambda)$ and $s \in CLF(G_4, \lambda)$, then $r+s \in CLF(G_3+G_4, \lambda)$.
Proof. Proof is straightforward and so it is left. □

Lemma 2.5 Let G be a simple d-regular graph on $2n$ vertices which is 1-factorable. If $r \in C(d, \lambda)$, then $rn \in CLF(G_4, \lambda)$.
Proof. Denote $X_1 = \{1, \ldots, d\}$ and $X_2 = \{d+1, \ldots, 2d\}$. Let $\mathcal{F} = \{F_1, \ldots, F_d\}$ be a completely decomposable λ-factorization with support size r of $\lambda K_{d,d}$ on bipartition (X_1, X_2). Since $\mathcal{F}$ is completely decomposable, we can find λ 1-factorizations of $K_{d,d}$ on bipartition (X_1, X_2), e.g. $\mathcal{F}_i = \{F^i_1, \ldots, F^i_d\}$, such that $F_i = \sum_{j=1}^{\lambda} F^j_i$, for $1 \le i \le d$. Now, for every $1 \le l \le \lambda$ $\mathcal{F}_l$ is a 1-factorization of $K_{d,d}$ on bipartition (X_1, X_2), thus there exists a Latin square $A^l = (a^l_{ij})$ of order d such that

$$F^l_i = \{\{a^l_{ij}, j+d\} | 1 \le j \le d\}, \quad 1 \le i \le d.$$

Now let $\{K_1, \ldots, K_d\}$ be any 1-factorization of G and for every $1 \le l \le \lambda$, define

$$H^l_{ij} = K_{a^l_{ij}}, \quad 1 \le i, j \le d.$$

Now, by Lemma 2.1 for every $1 \le l \le \lambda$, $\mathcal{H}_l = (H^l_{ij})$ is a Latin 1-factorization of G. Thus if we set

$$H_{ij} = \sum_{l=1}^{\lambda} H^l_{ij}, \quad 1 \le i, j \le d,$$

then $\mathcal{H} = (H_{ij})$ is a completely decomposable Latin λ-factorization of λG, and it is easy to see that its support size is rn. □

2.2 Necessary Conditions

In this section we obtain some necessary conditions on the support sizes of Latin λ-factorization. To do this, we first obtain some necessary conditions on the support sizes of

λ-factors and λ-factorizations of λG, where G is a simple 1-factorable graph of degree d on $2n$ vertices. The first lemma is trivial.

Lemma 2.6 Let F be a λ-factor of λG with support size t, then $t \in \{n, \ldots, \min\{\lambda, d\}.n\} \setminus \{n+1\}$, and if $\lambda = d$, then $t \neq nd - 1$. $\square$

The following lemma is proved in [3].

Lemma 2.7 Let G be a simple d-regular graph on $2n$ vertices which is 1-factorable. Then $CS(G, \lambda) \subseteq \{m, \ldots, M\} \setminus A$, where $m = nd$ and $M = \min\{\lambda, d\}.m$ and $A = \{m+1, m+2, m+3, m+5\}$ if $\lambda \neq d$ and $A = \{m+1, m+2, m+3, m+5, M-1, M-2, M-3, M-5\}$ otherwise. $\square$

Now, we deal with the Latin λ-factorizations of λG. Our goal is to prove the following lemma.

Lemma 2.8 Let $\mathcal{F} = \{F_1, \ldots, F_d\}$ be any Latin λ-factorization of λG, and let r be its support size. Then $r \in \{d^2n, \ldots, \min\{\lambda, d\}.d^2n\} \setminus \{d^2n + i | i = 1, \ldots, 7, 9, 10, 11, 13\}$, and if $\lambda = d$, then $r \notin \{d^3n - i | i = 1, \ldots, 7, 9, 10, 11, 13\}$.

Proof. By definition, for every $1 \leq i \leq d$, $\mathcal{F}_i = \{F_{ij} \mid 1 \leq j \leq d\}$ and $\Gamma_i = \{F_{ij} \mid 1 \leq j \leq d\}$ are two λ-factorizations of λG. For every i, j, $1 \leq i, j \leq d$, let r_{ij} denote the support size of F_{ij} and let r_i and s_j denote the support sizes of $\mathcal{F}_i$ and Γ_j, respectively. Clearly, we have $r = \sum_{i=1}^{d} r_i = \sum_{j=1}^{d} s_j = \sum_{i,j=1}^{d} r_{ij}$.

By Lemma 2.6, we have $n \leq r_{ij} \leq \min\{\lambda, d\}.n$, $r_{ij} \neq n+1$, and if $\lambda = d$, then $r_{ij} \neq dn - 1$, and hence $d^2n \leq r \leq \min\{\lambda, d\}d^2n$. Let $r \neq d^2n$, and hence for some i and j, $r_i > dn$ and $s_j > dn$. Without loss of generality, we can suppose that for some k, l, $1 \leq k, l \leq d$, we have $r_i > dn$ if and only if $i \leq k$, and $s_j > dn$ if and only if $j \leq l$. Also let $A = \{\ \{i, j\} \mid r_{ij} \neq n\}$, and $m = |A|$. Clearly $k \geq 2$ and $l \geq 2$. If $k = 3$ or $l = 3$, then by Lemma 2.7, $r > d^2n + 12$ and $r \neq d^2n + 13$. If $k = l = 2$, then $A = \{\ \{1,1\},\ \{1,2\},\ \{2,1\},\ \{2,2\}\ \}$. Now, it is immediately seen that $r_{11} = r_{12} = r_{21} = r_{22}$. Thus $r = (d^2 - 4)n + 4r_{11}$, and since $r_{11} > n + 1$, the assertion holds. A similar argument shows that if $\lambda = d$, then $r \notin \{d^3n - i | i = 1, \ldots, 7, 9, 10, 11, 13\}$. $\square$

2.3 Complete Graphs

Let $A(n, d, \lambda)$ denote the set of all integers k such that there exists a simple d-regular bipartite graph G on $2n$ vertices which is 1-factorable and $k \in CLF(G, \lambda)$, and let $B(n, d, \lambda)$

denotes the set of all integers k such that there exists a simple d-regular graph G on $2n$ vertices such that both G and $\overline{G}$ are 1-factorable, and $k \in CLF(G,\lambda)$. In this section, we completely determine $A(n,d,\lambda)$ and $B(n,d,\lambda)$ for $d \geq 6$.

For any three positive integers n, d, and λ, define

$$AS(n,d,\lambda) = \begin{cases} \{m,\ldots,M\} \setminus A, & \text{if } \lambda \neq d, \\ \{m,\ldots,M\} \setminus (A \cup B), & \text{otherwise,} \end{cases}$$

where $m = nd^2$, $M = \min(d,\lambda)m$, $A = \{m+i | i \in C\}$, $B = \{M - i | i \in C\}$, $C = \{1,\ldots,7,9,10,11,13\}$. The main results of this section are the two following theorems:

Theorem 1 Let n and d be two positive integers such that $d \leq n$. If $d = 6$ and $2d \leq n$ or $d \geq 12$, then $A(n,n,\lambda) = AS(n,d,\lambda)$.

Theorem 2 Let n and d be two positive integers such that $d \leq 2n-1$. If $12 \leq d$, then $B(n,d,\lambda) = AS(n,d,\lambda)$.

To prove these theorems, we develop some methods to determine $A(n,d,\lambda)$ and $B(n,d,\lambda)$ from $A(m,d_1,\mu)$. In this way, the following lemma is our main tool.

Lemma 2.9 Let A and B be two disjoint n-set and let $X = A \cup B$. Let $d \leq n$ and let G be a simple d-regular bipartite graph on bipartition (A,B) which is 1-factorable. Then (i) $K_{n,n} \setminus G$ is 1-factorable, and (ii) if either $d < n$ or $n \equiv 0 \pmod 2$, then $\overline{G} = K_{2n} \setminus G$ is also 1-factorable.

Proof. Part (i) is an immediate consequence of the well known fact that every regular bipartite graph is 1-factorable, and for part (ii) note that if $n \equiv 0 \pmod 2$, then $\overline{K_{n,n}}$ is the union of two vertex-disjoint copies of K_n and so it is 1-factorable, and for odd n it is easy to see that the complement of a $(n-1)$-regular bipartite graph on $2n$ vertices (which is unique up to isomorphism) is 1-factorable. □

Lemma 2.10 Let d, d_1 and m be three positive integers such that $4 \leq 2d_1 \leq d \leq m$, and denote $k = d - d_1$. If $r \in A(m,d_1,\mu)$, $s \in S_p(d_1,\mu)$, $2 \leq p \leq \min(d_1,\mu)$, and $q \in \{0,\ldots,k\} \setminus \{1,k-1\}$, then $r + skm + (kp-q)md + tmd^2 \in A(m,d,\mu+t)$ for $t = 0,\ldots,k$.

Proof. Denote $A = \{1,\ldots,n\}$ and $B = \{m+1,\ldots,2m\}$. Let G_1 be a simple 1-factorable bipartite graph of degree d_1 on bipartition (A,B) such that $r \in CLF(G,\mu)$. Let $\{F_1,\ldots,F_m\}$ be a 1-factorization of $K_{m,m}$ on bipartition (A,B) such that $\{F_1,\ldots,F_{d_1}\}$ is a 1-factorization of G_1. Let $G = \cup_{i=1}^{d} F_i$. Then G is a simple d-regular bipartite graph on bipartition (A,B), and G_1 is a subgraph of G, and both G_1 and $G_2 = G \setminus G_1$ are 1-factorable. Now, the assertion follows from Lemma 2.3. □

Lemma 2.11 Let d, d_1 and m be three positive integers such that $4 \leq 2d_1 \leq d \leq 2m - 1$, and denote $k = d - d_1$. If $r \in A(m, d_1, \mu)$, $s \in S_p(d_1, \mu)$, $2 \leq p \leq \min(d_1, \mu)$, and $q \in \{0, \ldots, k\} \setminus \{1, k-1\}$, then $r + skm + (kp - q)md + tmd^2 \in B(m, d, \mu + t)$ for $t = 0, \ldots, k$.
Proof. Proof of this lemma is essentially similar to the proof of Lemma 2.10 and so it is left. □

In view of these two lemmas, to prove Theorems 1 and 2 we must partially determine $A(n, d, \mu)$ for small d's. The following lemma is an immediate consequence of Lemma 2.4.

Lemma 2.12 If $r_1 \in A(n_1, d, \lambda)$ and $r_2 \in A(n_2, d, \lambda)$, then $r_1 + r_2 \in A(n_1 + n_2, d, \lambda)$. □

Lemma 2.13 If n is a positive integer greater than 4, then

$$\{4l | n \leq l \leq \min\{2, \lambda\} n\} \setminus \{4n + 4, 8n - 4\} \subset A(n, 2, \lambda)$$

Proof. By Lemma 2.5 we have $\{8, 16\} \subset A(2, 2, \lambda)$ and $\{12, 24\} \subset A(3, 2, \lambda)$ for every $2 \leq \lambda$. Now the result follows by induction on n (and utilizing Lemma 2.12). □

Lemma 2.14 Let $n \geq 6$. If $r_1 \in A(n, 2, \lambda)$, $r_2, r_3 \in \{4n, 8n\}$, $k \in \{0, 2, 4\}$, and $0 \leq u \leq 4$, then $b = \sum_{i=1}^{3} r_i + 6(8 - k)n + 36un \in A(n, 6, \lambda + u)$.
Proof. Denote $A = \{1, \ldots, n\}$ and $B = \{n + 1, \ldots, 2n\}$. Let G_1 be a simple 2-regular bipartite graph on bipartition (A, B) such that $r \in CLF(G_1, \lambda)$. Let $\{F_1, \ldots, F_n\}\}$ be a 1-factorization of $K_{n,n}$ on bipartition (A, B) such that $\{F_1, F_2\}$ is a 1-factorization of G_1. For $i = 2, 3$, let $G_i = F_{2i-1} + F_{2i}$, and let $G = \sum_{i=1}^{3} G_i$. We must show that $b \in CLF(G, \lambda + u)$. Let $A = (a_{ij})$ be any Latin square of order 6 whose first two rows are

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{matrix}$$

and define $K_{ij} = F_{a_{ij}}$ for $1 \leq i, j \leq 6$. It is not difficult to see that there exist a permutation σ on $\{1, \ldots, 2d\}$ with exactly $(2k + 2)$ fixed points such that $\sigma(1) = 1$, $\sigma(2) = 2$, and $\sigma(i) \in \{a_{1i}, a_{2i}\}$ for $i = 1, \ldots, 2d$. For $1 \leq l \leq 3$ let (N^l_{ij}) be a completely decomposable Latin λ-factorization with support size r_l of λG_l. Now for $1 \leq i \leq 2$ define

$$M_{ij} = \begin{cases} N^l_{i1}, & \text{if } j = 2l - 1, \\ N^l_{i2}, & \text{if } j = 2l, \end{cases}$$

and for $3 \leq i \leq 6$ define $M_{ij} = K_{ij} + K_{\sigma(i)j}$. and let

$$H_{ij} = M_{ij} + \sum_{l=3}^{u+2} K_{\eta_l(i)j}, \quad 1 \leq i,j \leq 6,$$

in which $\eta_l(i) = a_{il}$. Now it is easy to check that $\mathcal{L} = (L_{ij})$ is a completely decomposable Latin $(\lambda + u)$ -factorization with support size b of $(\lambda + u)G$. □

Lemma 2.15 If $n \geq 12$, then $A(n, 6, \lambda) = AS(n, 6, \lambda)$.

Proof. Denote $X_1 = \{1, \ldots, n\}$, $X_2 = \{x_1, \ldots, x_6\}$. $A \cap B = \emptyset$. Denote $m = n - 6$. By Lemmas 2.13 and 2.14 we have

$$\{4k | 9m \leq k \leq \min\{\lambda, 6\}.9m\} \setminus \{4ln + 4,\ 4ln - 4 | 9 \leq l \leq 9\min(\lambda, 6)\} \subset A(m, 6, \lambda).$$

In [2], it is shown that

$$\{216, 224, 228, 230, \ldots, 235, 237, 431\} \subset A(6, 6, 2).$$

Also, it is easy to show that

$$\{216l - i | 1 \leq i \leq 7\} \subset A(6, l, l), \quad \text{for } 2 \leq l \leq 5,$$
$$\{1289, \ldots, 1295\} \subset A(6, 7, 7).$$

Now, the assertion is a straightforward consequence of Lemma 2.4. □

Proof of Theorem 1 In view of Lemma 2.10, it is easy to see that if the assertion is true for $d = d_1$ (where $d_1 = 6$ or $d_1 \geq 12$) then it is also true for $d \in \{2d_1, \ldots, 4d_1 - 1\}$, and by Lemma 2.15 the assertion is true for $d = 6$ and $n \geq 12$. Now, the result follows by induction on d. □

Proof of Theorem 2 The assertion is a straightforward consequence of Lemmas 2.9, 2.12 and Theorem 1. □

3 Semi-Latin Factorizations of Complete Graphs

In this section, we develop some recursive methods to construct semi-Latin factorization with different support sizes, and then utilizing them, we essentially determine $SLF(K_{4n}, \lambda)$ for $n \geq 6$.

Let k be a positive integer and $k \equiv 0 \pmod 2$. Let G be a simple $(2k-1)$-regular graph on $2n$ vertices which is 1-factorable. Let G_1 be a k-regular subgraph of G such that both G_1 and $G_2 = G \setminus G_1$ are 1-factorable.

Lemma 3.1 If $r_1, r_2 \in SLF(G_2, \lambda)$, $s_1 \in LF(G_2, k-1)$, $r_3 \in LF(G_1, \lambda)$, $s_2 \in LF(G_1, \lambda+1)$, and $b = s_1 + (2k-1)(k-1)n + k^2(k-1)n$.Then

(i) $r_1 + r_2 + r_3 \in SLF(G, \lambda)$,

(ii) $b + nk^2 \in SLF(G, k)$,

(iii) $b + r_1 + r_2 + s_2 \in SLF(G, k + \lambda)$.

Proof. Let $F = (F_{ij})$ and $N = (N_{ij})$ be two semi-Latin λ-factorizations with support size r_1 and r_2 of λG_2,respectively. Let $H = (H_{ij})$ be a Latin $(k-1)$-factorization with support sizes s_1 of $(k-1)G_2$, and let $K = (K_{ij})$ be a Latin λ-factorization with support size r_3 of λG_1. Let $L = (L_{ij})$ be a Latin $(\lambda+1)$-factorization with support size s_2 of $(\lambda+1)G_1$, and let $M = (M_{ij})$ be a Latin 1-factorization of G_1.

To prove (i), for every i and j, $1 \le i, j \le k$, let

$$\begin{aligned} A_{ij} &= F_{ij}, \\ A_{k+i,k+j} &= N_{ij}, \\ A_{i,k+j} &= K_{ij}. \end{aligned}$$

Then $A = (A_{ij})$ is a semi-Latin λ-factorization with support size $\sum_{i=1}^{3} r_i$ of λG.

(ii) Let

$$\begin{aligned} &B_{ij} = B_{k+i,k+j} = G_1, && 1 \le i < j \le k, \\ &B_{j,k+1} = G_2, && 1 \le j \le k, \\ &B_{i,k+1+j} = H_{ij}, && 1 \le i, j \le k-1, \\ &B_{k,j} = G_2, && k+1 \le j \le 2k. \end{aligned}$$

Now if $1 \le i \le k < j \le d$, then B_{ij} is a $(k-1)$-factor of kG_1 (and consequently of kG) and otherwise it is a k-factor of kG_2 (and consequently of kG). Also, it is easy to check that

$$\sum_{j=1}^{2k} B_{ij} = (k-1)G_1 + kG_2, \qquad \text{for } 1 \le i \le k,$$

and

$$\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} |B^*_{ij}| = (k^2 + 2k - 1)(k-1)n + s_1 = b.$$

Thus, if we let

$$C_{ij} = \begin{cases} B_{ij} + M_{i(j-k)}, & \text{if } 1 \le i \le k < j \le 2k, \\ B_{ij}, & \text{otherwise,} \end{cases}$$

then (C_{ij}) would be a semi-Latin k-factorization with support size $b + nk^2$ of kG. (Note that if $1 \le i < k < j \le 2k$, then B_{ij} is a subgraph of kG_2, and consequently B_{ij} and $M_{i(j-k)}$ are edge-disjoint.) This proves (ii).

To Prove (iii), let

$$D_{ij} = \begin{cases} B_{ij} + F_{ij}, & 1 \le i < j \le k, \\ B_{ij} + L_{ij}, & 1 \le i \le k < j \le 2k, \\ B_{ij} + G_{ij}, & k < i < j \le 2k \end{cases}$$

Now, it is not difficult to see that (D_{ij}) is a semi-Latin $(k+\lambda)$-factorization of $(k+\lambda)G$ and its support size is

$$\begin{aligned} \textstyle\sum_{i<j} |D^*_{ij}| &= \textstyle\sum_{i,j=1}^{k}(|B^*_{ij}| + |F^*_{ij}|) + \sum_{i=1}^{k}\sum_{j=k+1}^{2k}(|B^*_{ij}| + |L^*_{ij}|) + \sum_{k<i<j\le 2k}(|B^*_{ij}| + |G^*_{ij}|) \\ &= b + r_1 + r_2 + s_2. \quad \square \end{aligned}$$

Lemma 3.1 presents an inductive method to determine the spectrum of support sizes of semi-Latin factorizations of a multigraph G from the spectrum of its subgraphs. Thus, to apply this lemma, we need a partial determination of $SLF(G,\lambda)$ for any simple 1-factorable graph G (of odd degree). The following lemma gives such a determination.

Lemma 3.2 Let G_2 be a simple $(2k-1)$-regular graph which is 1-factorable. If $r \in CS(G_2, \lambda, \lambda)$, then $kr \in SLF(G_2, \lambda)$.

Proof. Let $\{H_1, \ldots, H_{2k-1}\}$ be a λ-factorization with support size r of λG_2 and let $A = (a_{ij})$ be a symmetric Latin square of order $2k$ on $\{0, \ldots, 2k-1\}$ such that $a_{ii} = 0$ for $0 \le i \le 2k-1$. For $1 \le i < j \le 2k$, let $F_{ij} = H_{a_{ij}}$. It is easy to see that (F_{ij}) is a semi-Latin λ-factorization with support size kr of λG_2. $\square$

Lemma 3.3 Let n be a positive integer such that $n \equiv 0 \pmod 2$ and $n \ge 12$. If $36n - i \in A(n, 6, 2)$, then $n^3(2n-1) - i \in SLF(K_{2n}, n)$.

Proof. Let G_1 be a simple 6-regular bipartite graph on $2n$ vertices such that $36n - i \in CLF(G_1, 2)$. By Lemma 2.9, there exists a 1-factorization $\{F_1, \ldots, F_n, H_1, \ldots, H_{n-1}\}$ of K_{2n} such that $\{F_1, \ldots, F_6\}$ is a 1-factorization of G_1. Let

$$\begin{aligned} G_2 &= \textstyle\sum_{i=1}^{n} F_i, \quad & G_3 &= \textstyle\sum_{i=1}^{n-1} F_i, \\ G_4 &= \textstyle\sum_{i=1}^{n-1} H_i, \quad & G_5 &= \textstyle\sum_{i=1}^{n} H_i, \end{aligned}$$

where $H_n = F_n$. Let $A = (a_{ij})$ be any Latin square of order n such that $1 \le a_{ij} \le 6$ if

$1 \leq i,j \leq 6$. Define n permutations $\sigma_1, \ldots, \sigma_n$ by the following rule:

$$\sigma_i(j) = a_{ij}, \ 1 \leq i,j \leq n.$$

Now, for every $1 \leq i,j \leq n$, define

$$K_{ij} = F_{a_{ij}} \text{ and } L_{ij} = H_{a_{ij}}.$$

Clearly, $\mathcal{K} = (K_{ij})$ and $\mathcal{L} = (L_{ij})$ are two Latin 1-factorizations of G_2 and G_5, respectively. Let $\mathcal{N} = (N_{ij})$ be any Latin 2-factorization with support size $36n - i$ of $2G_1$, and for $1 \leq i,j \leq n$, define

$$B_{ij} = \begin{cases} N_{ij}, & \text{if } 1 \leq i,j \leq 6, \\ K_{\sigma_1(i)j} + K_{\sigma_2(i)j}, & \text{otherwise,} \end{cases}$$
$$C_{ij} = \textstyle\sum_{l=3}^{n} L_{\sigma_l(i)j}.$$

Clearly, (B_{ij}) is a Latin 2-factorization with support size $2n^2 - i$ of $2G_2$, and (C_{ij}) is a Latin $(n-2)$-factorization with support size $(n-2)n^2$ of $(n-2)G_5$. Now, we prove that for every $1 \leq i,j \leq n$, B_{ij} and C_{ij} are edge-disjoint. First note that $\{a_{\sigma_l(i)j} \mid 1 \leq l \leq n\} = \{1, \ldots, n\}$. Choose k such that $a_{\sigma_k(i)j} = n$. If $k \leq 2$, then C_{ij} is a subgraph of G_4 while B_{ij} is a subgraph of $2G_2$, and if $k \geq 3$, then B_{ij} is a subgraph of $2G_3$ while C_{ij} is a subgraph of G_4. Therefore, for every $1 \leq i,j \leq n$, B_{ij} and C_{ij} are edge-disjoint. Now, let (D_{ij}) be any semi-Latin factorization with support size $n^2(n-1)$ of $2G_4$, and let (E_{ij}) be a semi-Latin $(n-2)$-factorization with support size $n(n-1)(n-2)/2$ of $(n-2)G_3$ and define

$$\begin{aligned} M_{ij} &= D_{ij} + E_{ij} & & 1 \leq i < j \leq n, \\ M_{i(n+j)} &= B_{ij} + C_{ij} & & 1 \leq i, \ j \leq n, \\ M_{(n+i)(n+j)} &= M_{ij}, & & 1 \leq i < j \leq n. \end{aligned}$$

It is straightforward to check that $\mathcal{M} = (M_{ij})$ is a semi-Latin n-factorization with support size $n^3(2n-1) - i$ of nK_{2n}. $\square$

Now, we can prove our main result concerning semi-Latin factorizations. To do this, we need the following notation. For every n and λ, define

$$BS(m, \lambda) = \begin{cases} \{m, \ldots, M\} \setminus A, & \text{if } \lambda \neq n-1, \\ \{m, \ldots, M\} \setminus (A \cup B), & \text{otherwise,} \end{cases}$$

where $m = n^2(n-1)/2$, $M = \min\{\lambda, n-1\}.m$, $A = \{m+i | i = 1, \ldots, 7, 9, 10, 11, 13\}$, and $B = \{M - i | i = 1, \ldots, 7, 9, 10, 11, 13\}$.

Theorem 3 If $n \geq 6$, then $BS(4n, \lambda) \subseteq SLF(K_{4n}, \lambda)$.

Proof. Denote $X_1 = \{1, \ldots, 2n\}$, $X_2 = \{x_1, \ldots, x_{2n}\}$, $X_1 \cap X_2 = \emptyset$, and $X = X_1 \cup X_2$. We denote by G and G_1 the complete graph K_{4n} on X and complete bipartite graph $K_{2n,2n}$ on bipartition (X_1, X_2), respectively. Let $k = 2n$, and $G_2 = G \setminus G_1$. Clearly, G_2 is the union of two copies of K_{2n} on two disjoint set of vertices. Thus G_2 is 1-factorable. Now, the assertion is a straightforward consequence of Lemmas 3.1, 3.2, and 3.3 and Theorem 1. □

4 Support sizes of quadruple systems

In this section, we develop some recursive methods to construct quadruple systems with different support sizes, and then utilizing them we essentially determine the set $QSS(v, \lambda)$ for $v \equiv 0 \pmod 8$. Our main tools are some doubling constructions which enable us to construct a $QS(2u, \lambda)$ from two $QS(u, \lambda)$ on two disjoint sets of points.

Let v be an even integer greater than 6. Let $X_1 = \{1, \ldots, v\}$, $X_2 = \{x_1, \ldots, x_v\}$, $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$.

Lemma 4.1 Let $r_1, r_2 \in QSS(u, \lambda)$, and $r_3 \in SLF(u, \lambda)$. Then $\sum_{i=1}^{3} r_i \in QSS(2u, \lambda)$.

Proof. Let $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ be two $QS(u, \lambda)$ with support sizes r_1 and r_2, respectively. Let (F_{ij}) be a semi-Latin λ-factorization with support size r_3 of λK_u on X_2, and let

$$\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \sum_{1 \leq i < j \leq u} \{i, j\} * F_{ij}.$$

Then $(X, \mathcal{B})$ is a $QS(2u, \lambda)$ with support size $\sum_{i=1}^{3} r_i$. □

Lemma 4.2 Let u be a positive integer such that $u \geq 14$. If $r_1 \in SLF(u, \lambda + 2)$, $r_2, r_3 \in QSS(u, \lambda)$, $s \in SLF(u, 2)$, and $0 \leq l \leq u/2$, then

(i) $b_1 = u.m_{2u} - u^2(u-1) - 2lu(u-1) + s \in QSS(2u, u)$,

(ii) $b_2 = u.m_{2u} - u^2(u-1) - 2lu(u-1) + \sum_{i=1}^{3} r_i \in QSS(2u, u + \lambda)$.

Proof. Let $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$ be two $QS(u, \lambda)$ with support sizes r_1 and r_2, respectively. Let (F_{ij}) be a semi-Latin $(\lambda + 2)$-factorization with support size r_3 of $(\lambda + 2)K_u$ and let (G_{ij})

be a semi-Latin 2-factorization with support size s of $2K_u$. Let

$$\begin{aligned}
\mathcal{B}_0 &= P_4(X) \setminus (P_4(X_1) \cup P_4(X_2) \cup P_2(X_1) * P_2(X_2)),\\
\mathcal{B}_3 &= \mathcal{B}_0 + \textstyle\sum_{1\leq i<j\leq u}\{i,j\} * G_{ij},\\
\mathcal{B}_4 &= \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2 + \textstyle\sum_{1\leq i<j\leq u}\{i,j\} * F_{ij}.
\end{aligned}$$

Then $(X, \mathcal{B}_3)$ is a $QS(2u,u)$ with support size $u.m_{2u} - u^2(u-1)+s$, and $(X,\mathcal{B}_4)$ is a $QS(2u, u+\lambda)$ with support size $u.m_{2u} - u^2(u-1) + \sum_{i=1}^{3} r_i$. This proves the assertion for $l = 0$. To prove the assertion for $l > 0$, let $(X_2, \mathcal{C}_1)$ and $(X_2, \mathcal{C}_2)$ be two disjoint simple triple systems of order u and index 6 (since $u \geq 14$, this is possible), and let

$$\begin{aligned}
\Gamma_1 &= \textstyle\sum_{j=1}^{l}\{2j\} * \mathcal{C}_1 + \sum_{j=1}^{l}\{2j-1\} * \mathcal{C}_2,\\
\Gamma_2 &= \textstyle\sum_{j=1}^{l}\{2j-1\} * \mathcal{C}_1 + \sum_{j=1}^{l}\{2j\} * \mathcal{C}_2.
\end{aligned}$$

Now, it is easy to see that Γ_1 and Γ_2 are disjoint; the number of occurences of each $T \in P_3(X)$ in Γ_1 and Γ_2 are the same; and each quadruple of Γ_1 is a nonrepeated quadruple of both of $\mathcal{B}_3$ and $\mathcal{B}_4$. Thus $(X, \mathcal{B}_3 \setminus \Gamma_1 + \Gamma_2)$ is a $QS(2u, u)$ with support size b_1, and $(X, \mathcal{B}_4 \setminus \Gamma_1 + \Gamma_2)$ is a $QS(2u, u+\lambda)$ with support size b_2. □

Let $u \equiv 2$ or $4 \pmod 6$. Let $A = (a_{ij})$ be any Latin square of order u which has no Latin subsquare of order 2. Without loss of generality, we can suppose that $a_{1i} = i$, for $i = 1, \ldots, u$. For every i and j, $(1 \leq i,j \leq u)$ define

$$b_{ij} = k \Longleftrightarrow a_{ik} = j.$$

It is easy to see that $B = (b_{ij})$ is also a Latin square of order u. Let $(X_1, \mathcal{B})$ be any Steiner quadruple system of order u, and for every $1 \leq u \leq u$ define

$$\begin{aligned}
R_u &= \{\{i,j,k,x_{b_{ul}}\} \mid \{i,j,k,l\} \in \mathcal{B}\},\\
S_u &= \{\{x_i,x_j,x_k,a_{ul}\} \mid \{i,j,k,l\} \in \mathcal{B}\},\\
T_u &= \{\{a_{ui},a_{uj},x_i,x_j\} \mid 1 \leq i,j \leq u\},\\
\mathcal{B}_u &= R_u + S_u + T_u.
\end{aligned}$$

Then $(X, \mathcal{B}_u)$'s $(1 \leq u \leq u)$ are u mutually disjoint Steiner quadruple systems of order $2u$ [5]. Thus, if we define

$$\Gamma = P_4(X) \setminus (\cup_{i=1}^{u} \mathcal{B}_i),$$

then (X, Γ) is a simple $QS(2u, u-3)$ and $P_4(X_1) \cup P_4(X_2) \subset \Gamma$.

Lemma 4.3 Let $u \equiv 2$ or $4 \pmod 6$. If $r_1, r_2 \in QSS(u, u-3)$, and $1 \le k \le \min\{\mu, u\}$, then $b = (u-3)q_{2u} - 2\binom{u}{4} + r_1 + r_2 + k.q_{2u} \in QSS(2u, u-3+\mu)$.

Proof. Let (X_1, Γ_1) and (X_2, Γ_2) be two $QS(u, ul-3)$ with support sizes r_1 and r_2, respectively. Let

$$\Gamma_3 = \Gamma \setminus (P_4(X_1) \cup P_4(X_2)),$$
$$\mathcal{B} = \textstyle\sum_{i=1}^{3} \Gamma_i + \sum_{i=1}^{k} \mathcal{B}_i + (\mu - k)\mathcal{B}_k.$$

Then $(X, \mathcal{B})$ is a $QS(2u, u-3+\mu)$ with support size b. □

Let $Y = \{1, \ldots, 8\}$. We define a 1-factorization of K_8 on Y as follows:

$$F_1 = \{\{1,2\}\{3,7\},\{4,8\},\{5,6\}\}$$
$$F_2 = \{\{1,3\}\{2,7\},\{4,6\},\{5,8\}\}$$
$$F_3 = \{\{1,4\}\{2,6\},\{3,8\},\{5,7\}\}$$
$$F_4 = \{\{1,5\}\{2,8\},\{3,6\},\{4,7\}\}$$
$$F_5 = \{\{1,6\}\{2,5\},\{3,4\},\{7,8\}\}$$
$$F_6 = \{\{1,7\}\{2,4\},\{3,5\},\{6,8\}\}$$
$$F_7 = \{\{1,8\}\{2,3\},\{4,5\},\{6,7\}\}.$$

Since $u \ge 16$, there exists a 1-factorizations $\{G_1, \ldots, G_{u-1}\}$ such that $F_i \subset G_i$, for $i = 1, \ldots, 7$ [6]. For every $1 \le i \le u-1$ define

$$K_i = \{\ \{x_k, x_l\}\ |\{k, l\} \in G_i\}.$$

Clearly, $\{K_1, \ldots, K_{u-1}\}$ is a 1-factorization of K_u on X_2. Let

$$L_i = \begin{cases} K_i + K_{i+1} & \text{for } 1 \le i \le u-2, \\ K_{u-1} + K_1 & \text{for } i = u-1. \end{cases}$$

Then, $\{L_1, \ldots, L_{u-1})$ is a 2-factorization of $2K_u$ on X_2. Let

$$\mathcal{C} = P_4(X) \setminus (\ X_1 * P_3(X_2) + X_2 * P_3(X_1) + \sum_{i=1}^{u-1} G_i * L_i).$$

Now, it is easy to check that $(X, \mathcal{C})$ is a simple $QS(2u, u-3)$ which contains $P_4(X_1) \cup P_4(X_2)$. Let $(X_1, \mathcal{B})$ be a Steiner quadruple system and define

$$\mathcal{B}_1 = \{\{i,j,k,x_l\},\{x_i,x_j,x_k,l\} | \{i,j,k,l\} \in \mathcal{B}\} \cup \{\{i,j,x_i,x_j\} | 1 \le i < j \le u\}.$$

Then, it is easy to see that $(X, \mathcal{B}_1)$ is a Steiner quadruple system of order $2u$ and $\mathcal{B}_1 \cap \mathcal{C} = \emptyset$. Let

$$\Gamma_1 = \mathcal{C} \cup \mathcal{B}_1,$$
$$\Gamma_2 = P_4(X) \setminus \Gamma_1.$$

Trivially, (X, Γ_1) and (X, Γ_2) are simple $QS(2u, u-2)$ and $QS(2u, u-1)$, respectively. By applying trade-off method on these two simple designs we can obtain some new values in $QSS(2u, u-2)$ and $QSS(2u, u-1)$ which are not obtained from Lemmas 4.1, 4.2, and 4.3.

Lemma 4.4 If $u \geq 16$, then $\{(u-2)q_u - i | i = 1, \ldots, 13\} \subset QSS(2u, u-2)$.

Proof. For $1 \leq i \leq 4$, we define two disjoint subsets T_{i1} and T_{i2} of $P_4(X)$ according to the following table:

i	T_{i1}				T_{i2}			
1	$36x_1x_2$	$36x_4x_8$	$37x_1x_4$	$37x_2x_8$	$36x_1x_4$	$36x_2x_8$	$37x_1x_2$	$37x_4x_8$
	$56x_1x_4$	$56x_2x_8$	$57x_1x_2$	$57x_4x_8$	$56x_1x_2$	$56x_4x_8$	$57x_1x_4$	$57x_2x_8$
2	$24x_2x_5$	$24x_3x_4$	$25x_2x_3$	$25x_4x_5$	$24x_2x_3$	$24x_4x_5$	$25x_2x_5$	$25x_3x_4$
	$34x_2x_3$	$34x_4x_5$	$35x_2x_5$	$35x_3x_4$	$34x_2x_5$	$34x_3x_4$	$35x_2x_3$	$35x_4x_5$
3	$23x_1x_7$	$23x_6x_8$	$24x_1x_6$	$24x_7x_8$	$23x_1x_6$	$23x_7x_8$	$24x_1x_7$	$24x_6x_8$
	$35x_1x_6$	$35x_7x_8$	$45x_1x_7$	$45x_6x_8$	$35x_1x_7$	$35x_6x_8$	$45x_1x_6$	$45x_7x_8$
4	1234	1256	1357	1467	1235	1246	1347	1567
	2348	2568	3578	4678	2358	2468	3478	5678

It is an easy exercise to check that (i) if $(i,j) \neq (k,l)$, then $T_{ij} \cap T_{kl} = \emptyset$, (ii) for every $1 \leq i \leq 4$ the number of occurences of each $T \in P_3(X)$ in T_{i1} and T_{i2} are the same, and (iii) for every $1 \leq i \leq 4$, $T_{i1} \subset \Gamma_1$ while

$$|T_{i2} \cap \Gamma_1| = 2^{i-1}, \text{ for } 1 \leq i \leq 4.$$

Hence, if we let $\mathcal{F}_0 = \Gamma_1$ and

$$\begin{aligned}
\mathcal{F}_1 &= (\mathcal{F}_0 \setminus T_{11}) + T_{12}, \\
\mathcal{F}_2 &= (\mathcal{F}_0 \setminus T_{21}) + T_{22}, \\
\mathcal{F}_3 &= (\mathcal{F}_1 \setminus T_{21}) + T_{22}, \\
\mathcal{F}_j &= (\mathcal{F}_{j-4} \setminus T_{31}) + T_{32}, \qquad j = 4, \ldots, 7 \\
\mathcal{F}_j &= (\mathcal{F}_{j-8} \setminus T_{41}) + T_{42}, \qquad 8 \leq j \leq 14
\end{aligned}$$

then for every $1 \leq j \leq 14$ $(X, \mathcal{F}_j)$ is a $QS(2u, u-2)$ with support size $(u-2)q_u - j$. $\square$

Lemma 4.5 If $u \geq 16$, then $\{(u-1)q_u - i | i = 1, \ldots, 13\} \subset QSS(2u, u-1)$.

Proof. For $1 \leq i \leq 6$, we define two disjoint subsets T_{i1} and T_{i2} of $P_4(X)$ according to the following table:

i	T_{i1}				T_{i2}			
1	$12x_1x_3$	$12x_5x_8$	$14x_1x_5$	$14x_3x_8$	$12x_1x_5$	$12x_3x_8$	$14x_1x_3$	$14x_5x_8$
	$26x_1x_5$	$26x_3x_8$	$46x_1x_3$	$46x_5x_8$	$26x_1x_3$	$26x_5x_8$	$46x_1x_5$	$46x_3x_8$
2	$37x_2x_7$	$37x_4x_6$	$38x_2x_6$	$38x_4x_7$	$37x_2x_6$	$37x_4x_7$	$38x_2x_7$	$38x_4x_6$
	$57x_2x_6$	$57x_4x_7$	$58x_2x_7$	$58x_4x_6$	$57x_2x_7$	$57x_4x_6$	$58x_2x_6$	$58x_4x_7$
3	$24x_2x_3$	$24x_4x_5$	$25x_2x_4$	$25x_3x_5$	$24x_2x_4$	$24x_3x_5$	$25x_2x_3$	$25x_4x_5$
	$34x_2x_4$	$34x_3x_5$	$35x_2x_3$	$35x_4x_5$	$34x_2x_3$	$34x_4x_5$	$35x_2x_4$	$35x_3x_5$
4	$16x_1x_7$	$16x_6x_8$	$17x_1x_8$	$17x_6x_7$	$16x_1x_8$	$16x_6x_7$	$17x_1x_7$	$17x_6x_8$
	$68x_1x_8$	$68x_6x_7$	$78x_1x_7$	$78x_6x_8$	$68x_1x_7$	$68x_6x_8$	$78x_1x_8$	$78x_6x_7$
5	$16x_2x_4$	$16x_3x_5$	$17x_2x_3$	$17x_4x_5$	$16x_2x_3$	$16x_4x_5$	$17x_2x_4$	$17x_3x_5$
	$68x_2x_3$	$68x_4x_5$	$78x_2x_4$	$78x_3x_5$	$68x_2x_4$	$68x_3x_5$	$78x_2x_3$	$78x_4x_5$
6	$24x_1x_8$	$24x_6x_7$	$25x_1x_7$	$25x_6x_8$	$24x_1x_7$	$24x_6x_8$	$25x_1x_8$	$25x_6x_7$
	$34x_1x_7$	$34x_6x_8$	$35x_1x_8$	$35x_6x_7$	$34x_1x_8$	$34x_6x_7$	$35x_1x_7$	$35x_6x_8$

It is an easy exercise to check that (i) if $(i,j) \neq (k,l)$, then $T_{ij} \cap T_{kl} = \emptyset$, (ii) for every $1 \leq i \leq 6$ the number of occurences of each $T \in P_3(X)$ in T_{i1} and T_{i2} are the same, and (iii) for every $1 \leq i \leq 6$, $T_{i1} \subset \Gamma_2$ while

$$|T_{12} \cap \Gamma_2| = |T_{22} \cap \Gamma_2| = 1,$$
$$|T_{32} \cap \Gamma_2| = |T_{42} \cap \Gamma_2| = 2,$$
$$|T_{52} \cap \Gamma_2| = |T_{62} \cap \Gamma_2| = 4.$$

Hence, if we let $\mathcal{F} = \Gamma_2$ and

$$\begin{aligned}
\mathcal{F}_1 &= (\mathcal{F} \setminus T_{12}) + T11, \\
\mathcal{F}_2 &= (\mathcal{F}_1 \setminus T_{21}) + T_{22}, \\
\mathcal{F}_j &= (\mathcal{F}_{j-2} \setminus T_{31}) + T_{32}, && j = 3, 4, \\
\mathcal{F}_j &= (\mathcal{F}_{j-4} \setminus T_{51}) + T_{52}, && 5 \leq j \leq 8, \\
\mathcal{F}_j &= (\mathcal{F}_{j-4} \setminus T_{61}) + T_{62}, && 9 \leq j \leq 12, \\
\mathcal{F}_j &= (\mathcal{F}_{j-2} \setminus T_{41}) + T_{42}, && 13 \leq j \leq 14.
\end{aligned}$$

Then for every $1 \leq j \leq 14$ $(X, \mathcal{F}_j)$ is a $QS(2u, u-1)$ with support size $(u-1)q_u - j$. □

In view of Lemmas 4.1-4.5, in order to determine $QSS(2u,\lambda)$ we only need a partial determination of $QSS(u,\lambda)$ for all λ. It is well known that if $v \equiv 4$ or $8 \pmod{12}$, then $v/2$ mutually disjoint Steiner quadruple systems exist [5] and if $v \equiv 0 \pmod 6$ then a large set of disjoint $QS(v,3)$'s exists[8]. Utilizing these facts one can easily establish two following lemmas.

Lemma 4.6 If $u \equiv 0 \pmod 6$ and $\lambda \equiv 0 \pmod 3$, then $\{3jq_u | 1 \le j \le \min\{t,(u-3)/3\}\} \subset QSS(u,3t)$. $\Box$

Lemma 4.7 If $u \equiv 4$ or $8 \pmod{12}$, then $\{jq_u | 1 \le j \le \min\{t, u-3\}\} \subset QSS(u,t)$. $\Box$

To apply Lemma 4.3 we need a partial determination for $QS(v, v-3)$ for all $v \equiv 4$ or $8 \pmod{12}$.

Lemma 4.8 If $u \equiv 2$ or $4 \pmod 6$ and $u \ge 28$, then $\{\binom{u}{4} - i | 14 \le i \le q_u\} \subset QSS(u,u-3)$.

Proof. Let $14 \le i \le q_u$ and let $(Y,\mathcal{B}_1)$ and $(Y,\mathcal{B}_2)$ be two Steiner quadruple systems of order u intersecting in i quadruples, and let $\mathcal{B} = P_4(Y) \setminus \mathcal{B}_1 + \mathcal{B}_2$. Then $(Y,\mathcal{B})$ is a $QS(u,u-3)$ with support size $\binom{u}{4} - i$. $\Box$

Lemma 4.9 Let $u \equiv 4$ or $8 \pmod{12}$, and $u \ge 24$. If $1 \le k \le u/4$ and $1 \le l \le (u-8)/4$, then $\binom{u}{4} - kl \in QSS(u,u-3)$.

Proof. Let $u = 2n$, $X_1 = \{1,\dots,n\}$, $X_2 = \{x_1,\dots,x_n\}$, $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$. Let $Y = \{x_2,\dots,x_n\}$. Let $(Y,\mathcal{B}_1)$ and $(Y,\mathcal{B}_2)$ be two disjoint simple $TS(n,l)$'s and define

$$\mathcal{B} = P_4(X) \setminus (\sum_{i=1}^{k}(2i) * \mathcal{B}_1 + \sum_{i=1}^{k}(2i-1) * \mathcal{B}_2) + \sum_{i=1}^{k}(2i) * \mathcal{B}_2 + \sum_{i=1}^{k}(2i-1) * \mathcal{B}_1.$$

Then $(X,\mathcal{B})$ is a $QS(u,u-3)$ with desired support size. $\Box$

The following lemma is proved in [2].

Lemma 4.10 If $v \equiv 0 \pmod 6$, and $v \ge 24$, then

$$\{t_v, t_v+8, t_v+12, t_v+14, \dots, v(v-1)(v-2)/12\} \subset QSS(v,3). \Box$$

Now, we state and prove our main results concerning $QSS(v,\lambda)$. let $v \equiv 0 \pmod 2$, and let $\lambda_v = \gcd(v-3,12)$, $q_v = v(v-1)(v-2)/12$, $t_v = v(v^2-3v+42)/6$, $M_v = \min\{\lambda, v-3.t_v\}$ and $A = \{1,\dots,7,9,10,11,13\}$. If there exists no quadruple system of order v and index λ, the set $PS(v,\lambda)$ of possible support sizes of $QS(v,\lambda)$'s is empty; and otherwise except for $\lambda = v-3$, we define $PS(v,\lambda)$ according to the following table

$v \pmod 6$	$PS(v,\lambda)$
0	$\{t_v,\ldots,M_v\} \setminus \{t_v+i \mid i \in A\}$
2, 4	$\{q_v,\ldots,M_v\} \setminus \{q_v+i \mid i \in A\}$

For $\lambda = v-3$, $PS(v,\lambda)$ is defined similarly, only with the omission of $\{M_v - i \mid i \in A\}$.

Theorem 4 If $v \equiv 0 \pmod 8$, and $v \geq 48$, then $PS(v,\lambda) \subset QSS(v,\lambda)$.

Proof. Let $u = v/2$, and $r \in PS(v,\lambda)$. If either $r \leq (u-2)q_v - 2q_u$ or $r \leq (u-1)q_v - 4q_u - 14$ and $\lambda \geq u-1$, then due to Lemma 4.1, $r \in QSS(v,\lambda)$. If $r \geq (u-2)q_v - 2q_u$ and $\lambda \geq u$, then due to Lemma 4.2 $r \in QSS(v,\lambda)$. If either $(u-2)q_v - 2q_u \leq r \leq (u-2)q_v - 14$ and $\lambda = u-2$ or $(u-1)q_v - 4q_u - 14 \leq r \leq (u-1)q_v - 4q_u$ and $\lambda = u-1$ (note that in both cases we have $u \equiv 4$ or $8 \pmod{12}$), then due to Lemma 4.3 $r \in QSS(v,\lambda)$. Finally if $\lambda \in \{u-2, u-1\}$ and $\lambda q_v - 14 \leq r \leq \lambda q_v$, then due to Lemmas 4.5 and 4.6 $r \in QSS(v,\lambda)$. □

Concluding Remarks

1. Simple counting arguments show that there is no $(3,4,v)$ trade of volume i for $i \in \{1,\ldots,7,9,10,11,13\}$, which in turn implies there exist no $QS(v,v-3)$ with support size r for $r \in \{1,\ldots,7,9,10,11,13\}$.

2. Let $v \equiv 2$ or $4 \pmod 6$, and let $(X,\mathcal{B})$ be a $QS(v,\lambda)$ with support size $b > q_v$. For $i \in X$, let

$$\mathcal{B}_i = \{B \setminus \{i\} \mid i \in B \in \mathcal{B}\},$$

then for $i \in X$ $(X \setminus \{i\}, \mathcal{B}_i)$ is a triple system of order $v-1$ and index λ. Let $b_i = |\mathcal{B}_i^*|$. Then $b = (\sum_{i \in X} b_i)/4$. It is well known that $b_i \geq m_{v-1} = (v-1)(v-2)/6$ and $b_i \notin \{m_{v-1} + i \mid i = 1,2,3,5\}$. Also if either $b_i = m_{v-1} + 4$ or $b_i = m_{v-1} + 6$, then one can easily determine the structure of blocks of frequency less than λ up to isomorphism (in first case it is unique, and in the second case there are two possibilities). Putting these results together it is strightforward but tedious to show that

$$b \notin \{q_v + i \mid i = 1,\ldots,7,9,10,11,13\}.$$

Therefore for $v \equiv 8$ or $16 \pmod{24}$ we have

$$QSS(v,\lambda) = PS(v,\lambda).$$

3. Let $v \equiv 0 \pmod{24}$, and let $(X, \mathcal{B})$ be a $QS(v, \lambda)$ with support size b. It is well known that $b \geq t_v$, and again applying well known results on the support sizes of triple systems on derived designs one can show that $b \notin \{t_v + i | i = 1, \ldots, 5\}$. On the other hand it can be shown that $t_v + 6 \in QSS(v, \lambda)$. Therefore Theorem 4 determines $QSS(v, \lambda)$ with at least one and at most 6 possible omissions.

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