ON THE COLORABILITY OF GRAPHS DECOMPOSABLE INTO DEGENERATE GRAPHS WITH SPECIFIED DEGENERACY

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Abstract.

An m-degenerate graph is a graph, every subgraph of which has minimal degree at most m. An $(m_1, m_2, \ldots, m_s)$-composed graph is a graph, the edge set of which can be partitioned into s sets generating respectively graphs being $m_1, m_2, , \ldots, m_s$ degenerate. We conjecture that such a graph is $\sum_{i=1}^{s} m_i + \lfloor \frac{1}{2}(1 + \sqrt{\underset{1 \le i < j \le s}{1 + 8\Sigma m_i m_j}}) \rfloor$ colorable. Partial results are obtained, but not even Tarsi's case: $m_1 = 1$, $m_2 = 2$ is settled.

1. Introduction

The following two definitions of m-degenerate graphs have been formulated and shown to be equivalent in [8]. The same paper contains a study of the most elementary properties of m-degenerate graphs we shall use below. Also we mention [1] for further results on this class of graphs.

Definition 1. A graph G is said to be m-degenerate, for m a nonnegative integer, if every subgraph of G has minimum degree at most m.

Definition 2. A graph G is said to be m-degenerate if there is a labelling $v_1, v_2, \ldots, v_n$ of its vertices such that for $i = 1, 2, \ldots, n$ there are among the neighbors of v_i at most m vertices v_j with $j > i$. Call such edges outgoing.

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The following consequences of Definition 2 are also observed in [8] and [5].

Proposition 1. *If G is m-degenerate and has n vertices, then*

$$|E(G)| \leq mn - \frac{m(m+1)}{2} . \tag{1}$$

Proposition 2. *If G is m-degenerate, then G is $(m+1)$-colorable.*

In papers [4,5,6,7] we developed the concept of $(m_1, m_2, \ldots, m_s)$-composed graphs.

Definition 3. A graph G is said to be $(m_1, m_2, \ldots, m_s)$-composed if the edge set of G can be partitioned into the edge sets of graphs $M_1, M_2, \ldots, M_s$ being respectively $m_1, m_2, \ldots, m_s$ degenerate.

The main result of [5] is

Theorem 0. *K_n is $(m_1, m_2, \ldots, m_s)$-composed if and only if*

$$n \leq \sum_{i=1}^{s} m_i + \left\lfloor \frac{1}{2}\left(1 + \sqrt{1 + 8 \sum_{1 \leq i < j \leq s} m_i m_j}\right) \right\rfloor . \tag{2}$$

Denote the right side of (2) by $\nu(m_1, m_2, \ldots, m_s)$ and by v_s for short. To prove the only if part of Theorem 0, one uses the following generalization of Proposition 1. A constructive proof of the if part is given in [5].

Proposition 3. *If G is $(m_1, m_2, \ldots, m_s)$-composed and has n vertices then*

$$|E(G)| \leq n \sum_{i=1}^{s} m_i - \sum_{i=1}^{s} \frac{m_i(m_i+1)}{2} .$$

The generalization of Proposition 2 is difficult. The cases of $(1, m)$-composed and (m_1, m_2)-composed graphs were considered in [4] and [5] respectively. This paper is an attempt to establish the colorability of $(m_1, m_2, \ldots, m_s)$-composed graphs using tools and methods similar to those in [4] and [5].

2. Bounds

An obvious bound is established in the next proposition.

Proposition 4. *If G is $(m_1, m_2, \ldots, m_s)$-composed then it is $\prod_{i=1}^{s}(m_i+1)$ colorable.*

Proof: By Proposition 2 the graphs M_i are (m_i+1)-colorable and the cartesian product of the colorings will do.

A bound better in general can be obtained as a consequence of the following fact.

Proposition 5. *If G is $(m_1, m_2, \ldots, m_s)$-composed then G is $2\sum_{i=1}^{s} m_i - 1$ degenerate.*

Proof: One shows that every subgraph of G has a vertex of degree at most $2\sum_{i=1}^{s} m_i - 1$. This follows from the fact that, the average degree of an m degenerate graph is less than $2m$.

This gives immediately:

Proposition 6. *If G is $(m_1, m_2, \ldots, m_s)$-composed then G is $2\sum_{i=1}^{s} m_i$ colorable.*

Observe that the bound of Proposition 6 is never worse than the bound of Proposition 4 and is better except in the case $s = 2, m_1 = 1$, and $m_2 = m$. When $m = 1$ the bound is exact.

For every value of $m > 1$ it is not known whether the bound $2(1 + m)$ is exact. An interesting case is when $m = 2$. There are 5-chromatic $(1,2)$-composed graphs, for example K_5. The bound is 6, but it is still not known whether there exists $(1,2)$-composed graphs which are 6-chromatic. This question is due to Tarsi and raised in connection with [10].

Observe, that $\nu(1,2) = 5$.

In general by Theorem 0 there are $\nu(m_1, m_2, \ldots, m_s)$ chromatic $(m_1, m_2, \ldots, m_s)$-composed graphs and we close this section by conjecturing that a better bound than the bound of Proposition 6 can be obtained.

Conjecture 1. If a graph G is $(m_1, \ldots, m_s)$-composed then G is $\nu(m_1, m_2, \ldots, m_s)$-colorable.

The only case for which this is a theorem is $m_1 = m_2 = \cdots = m_s = 1$. Indeed, then $\nu(1, 1, \ldots, 1) = 2s$ and this equals the bound $2\sum_{i=1}^{s} m_i$. So we have

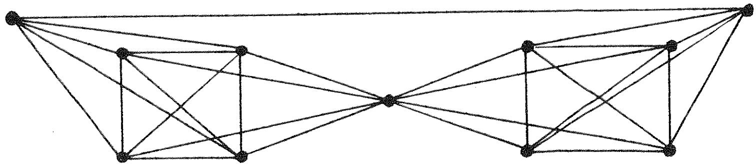
Theorem 1. *Any $(1, 1, \ldots, 1)$-composed graph is $2s$ colorable.*

3. Approach Based on Counting Edges

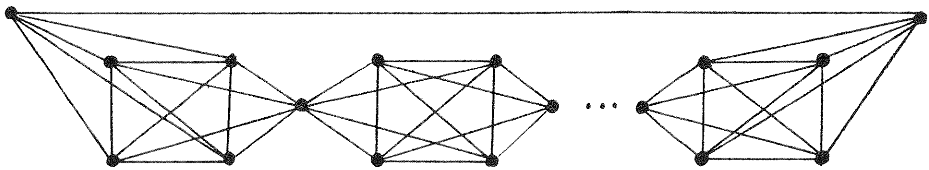
A natural way for proving Conjecture 1 would be to use the facts that a $(m_1, m_2, \ldots, m_s)$-composed graph has not too many edges and a $(\nu_s + 1)$-chromatic critical graph has not too few. This idea works for the complete graph K_{ν_s+1} but for more general $(\nu_s + 1)$-chromatic critical graphs it does not work.

Let us illustrate this by an example. Consider the case $m_1 = 1, m_2 = 2$. Then $\nu_2 = 5, K_6$ has 15 edges and this is more than a $(1,2)$-composed six vertex graph can have, namely 14, as shown in Proposition 3.

However, the 6-chromatic critical graph H on 11 vertices



has 29 edges and a $(1,2)$-composed graph on 11 vertices can have this many edges. Proposition 3 gives $3.11 - 4 = 29$. The graph H contains two blocks $K_6 - e$. Define H_t,



a similar graph containing t such blocks. H_t is also 6-chromatic and critical, has $5t + 1$ vertices and the number of its edges is less than a $(1,2)$-composed graph on $5t + 1$ vertices can have. On the other hand, we will show by other methods that H_t is not $(1,2)$-composed for any t.

For more general ν_s the situation is similar.

4. The Structural Approach

We shall describe some constructions of Hajós [3] and Ore [9] starting with $K_{\nu+1}$ which provide all graphs that are not ν-colorable.

Since K_{ν_s+1} is not $(m_1, \ldots, m_s)$-composed one could hope that non-composedness is preserved by the constructions.

4.1 Hajós's Construction. The following construction called conjunction is due to Hajós.

Definition 4. The conjunction G_0 of two disjoint graphs G_1 and G_2 is the graph obtained by deleting the edges $e_1 = (a_1, b_1)$, $e_2 = (a_2, b_2)$ of G_1 and G_2 respectively, identifying the vertices a_1 and a_2 to a single vertex a and adding a new edge (b_1, b_2).

One of the main results of this paper is the following, stating that the Hajós conjunction preserves the property of not being $(m_1, m_2, \ldots, m_s)$-composed.

Theorem 2. *If the graphs G_1 and G_2 are not $(m_1, m_2, \ldots, m_s)$-composed then their conjunction G_0 is also not $(m_1, m_2, \ldots, m_s)$-composed.*

Proof: Suppose the contrary, then for some j the edge (b_1, b_2) belongs to M_j and there is a labeling ι_j of M_j showing that M_j is m_j degenerate. Denote the graph $G_1 - e_1$ and $G_2 - e_2$ by G_1^- and G_2^- respectively. Then those graphs are $(m_1, m_2, \ldots, m_s)$-composed while G_1 and G_2 are not. Suppose without loss of generality that $\iota_j(b_1) < \iota_j(b_2)$ and that $\iota_j(a) > \iota_j(b_1)$. Then the edge (b_1, b_2) can be replaced by (b_1, a) contradicting the assumption on G_1. If $\iota_j(a) < \iota_j(b_1)$, observe that the number of outgoing edges from a is at most m_j so it cannot be m_j in both graphs G_1^- and G_2^-. Let this number be smaller in G_1^-. Then the edge (a, b_1) can be added having the same contradiction as above.

This result does not prove our conjecture since not every non-(ν_s)-colorable graph can be constructed in this way starting with K_{ν_s+1}'s. It proves, however, that our claim that the graph H_t introduced at the end of section 3 is not $(1,2)$-composed for any t, since H_t can be obtained by successive conjunctions of K_6's.

In order to obtain every not (ν_s+1)-colorable graph, one can use a construction of Ore called merger.

Definition 5. A merger of the disjoint graphs G_1 and G_2 is the graph G^0 obtained from G_0, the Hajós conjunction, by identifying $\alpha - 1$ additional pairs of vertices a', a'' $a' \in V(G_1 - a_1)$, $a'' \in V(G_2 - a_2)$ excluding the pair b_1, b_2, but not b_1, a'' or a', b_2. Denote the set of identified vertices by A.

If the number of pairs including a_1, a_2 is α the merger is called an α-merger. If $\beta \leq \alpha \leq \gamma$ it is called a $[\beta, \gamma]$-merger. If G^0 is obtained from K_ν's by applying successive mergers it is called a ν-amalgamation.

Ore proved that every graph which is not ν-colorable must contain a $(\nu+1)$-amalgamation, hence every critical $(\nu+1)$-chromatic graph is a $(\nu+1)$-amalgamation.

The statement generalizing Theorem 2 to mergers is not true. However, this does not disprove our conjecture 1 and we state the following equivalent conjecture.

Conjecture 2.

No $(\nu_s + 1)$-amalgamation is $(m_1, m_2, \ldots, m_s)$-composed. In particular for $m_1 = 1$, $m_2 = 2$, no 6-amalgamation is $(1,2)$-composed.

As a pessimistic observation we mention that by a theorem of Ore [9] the statement "No 5-amalgamation is planar" is equivalent to the 4-color theorem.

Although a merger does not preserve the property of not being $(m_1, \ldots, m_s)$-composed, in general, the property is preserved by α-mergers if α is not too big.

Theorem 3. *If G_1 and G_2 are not $(m_1, m_2, \ldots, m_s)$-composed then any α-merger G^0 of them is not $(m_1, \ldots, m_s)$-composed provided*

$$\alpha \leq \sum_{i=1}^{s} m_i \,. \tag{3}$$

Proof: First consider the case when the α-merger is b-free i.e. neither of b_1, b_2 occurs in any of the α pairs identified. The first part of the proof is showing as in the proof of Theorem 2 that in the labelling of M_j in G^0, j being the index such that (b_1, b_2) is an edge of M_j. $\iota_j(a)$ must be smaller than $\iota_j(b_1)$ and $\iota_j(b_2)$. Then observe that by (3), not for every i the number of outgoing edges from a and also from any other vertex in A can be m_i in both G_1 and G_2. Let j be the index with less than m_j edges outgoing in say G_1.

One can remove edges conveniently from some M_h to another M_k and have precisely for j less outgoing edges from a than m_j say in G_1. Then G_1 with (a_1, b_1) returning to it is $(m_1, m_2, \ldots, m_s)$-composed – a contradiction. It is not difficult to prove the non-b-free case.

Theorem 4. *If G is a $(\nu_s + 1)$-amalgamation obtained exclusively by $[1, \sum_{i=1}^{s} m_i]$-mergers, then G is not $(m_1, m_2, \ldots, m_s)$-composed*

Proof: This is a corollary of Theorem 3.

5. Combined Structural and Counting Method

Combining the counting and structural arguments, we shall establish a theorem similar to Theorem 4, but for α-mergers restricted to a different interval.

For this purpose, we introduce two definitions.

Definition 6. A graph G on n vertices with more edges than a $(m_1, m_2, \ldots, m_s)$-composed graph on n vertices can have (namely, $n \sum_{i=1}^{s} m_i - \sum_{i=1}^{s} \frac{m_i(m_i+1)}{2}$) will be called $(m_1, m_2, \ldots, m_s)$-redundant.

Definition 7. Define

$$\lambda(m_1, m_2, \ldots, m_s) = \sum_{i=1}^{s} m_i + \left\lceil \frac{1}{2}\left(1 - \sqrt{1 + 8 \sum_{1 \leq i < j \leq s} m_i m_j}\right)\right\rceil .$$

This will be denoted by λ_s for short.

Theorem 5. *If G_1 and G_2 are $(m_1, m_2, \ldots, m_s)$-redundant graphs then any $[\lambda_s, \nu_s]$-merger G of them is also $(m_1, m_2, \ldots, m_s)$-redundant.*

Proof: Let the number of vertices of G_1 and G_2 be respectively n_1 and n_2.

Suppose, contrary to the assertion in the theorem, that for some α-merger

$$|E(G)| \leq (n_1 + n_2 - \alpha) \sum_{i=1}^{s} m_i - \frac{1}{2} \sum_{i=1}^{s} m_i(m_i + 1) . \tag{4}$$

By the assumptions on G_1 and G_2 one has for $\ell = 1, 2$

$$(5) \qquad |E(G_\ell)| \geq \sum_{i=1}^{s} \left(n_\ell m_i - \frac{1}{2} \cdot m_i(m_i + 1) \right) + 1,$$

therefore

$$(6) \qquad |E(G)| > (n_1 + n_2) \sum_{i=1}^{s} m_i - 2 \sum_{i=1}^{s} \frac{m_i(m_i+1)}{2} - \frac{\alpha(\alpha - 1)}{2} .$$

From (4) and (6), one gets

$$\alpha^2 - (1 + 2 \sum m_i)\alpha + \sum m_i(m_i + 1) > 0 .$$

This contradicts the assumption that $\lambda_s \leq \alpha \leq \nu_s$.

Theorem 6. *If G is a $(\nu_s + 1)$-amalgamation obtained exclusively by $[\lambda_s, v_s]$-mergers then G is not $(m_1, m_2, \ldots, m_s)$-composed.*

Proof: This is a consequence of Theorem 5.

6. Final Remarks

The main results of this paper are Theorems 4 and 6. We mention here without proof some more results of the same kind which may help others to accomplish the proof of our conjectures.

Theorem 7. *If G_1 and G_2 are $(m_1, m_2, \ldots, m_s)$-redundant graphs then any b-free α-merger with $\alpha \geq \lambda_s$ contains an $(m_1, \ldots, m_s)$-redundant graph and therefore is not $(m_1, m_2, \ldots, m_s)$-composed.*

Theorem 8. *If each of G_1 and G_2 contain an $(m_1, \ldots, m_s)$-redundant graph then every b-free merger G of them is not $(m_1, m_2, \ldots, m_s)$-composed.*

Finally we mention two recent papers [2] and [11] dealing with more specific decompositions into degenerate graphs.

References

1. Bollobas, B., Extremal graph theory, Academic Press, London 1978, 222,280.
2. Fleischner, H. and Stiebitz M., A solution to a colouring problem of P. Erdös, Discrete Math. 101 (1992) 39-48.
3. Hajós, G., Über eine Konstruktion nicht n-färbbarer Graphen, Wissenchaftliche Zeitschrift der Martin Luther Universität, Halle-Wittenberg 1961, 116-117.
4. Klein, R., On the colorability of m-composed graphs, Discrete Mathematics 133 181-190 (1994).
5. Klein, R. and Schönheim, J., Decomposition of K_n into two graphs respectively m_1 and m_2 degenerated and Colorability of graphs having such a decomposition. Report No. 91/43 Combinatorics and its Application Report No. 91/107 Diskrete Structure in der Mathematic, Bielefeld University.
6. Klein, R. and Schönheim, J., Decomposition of K_n into degenerate graphs, In Combinatorics and Graph Theory Hefei, 6-27, April 1992. World Scientific. Singapore, New Jersey, London, Hong Kong, 141-155.
7. Klein, R., On $(m_1, m_2, \ldots, m_s)$-composed graphs, Thesis, Tel Aviv University 1992.
8. Lick, Don R. and White, A.T., k-Degenerate graphs, Canad. J. Math. vol. XXII, No.5 1970, 1082-1096.
9. Ore, D., The four color problem, Academic Press 1967.
10. Seymour, P.D., Nowhere-Zero G-flow, J. Combinatorical Theory, Ser B 30 1981, 130-135.
11. Stiebitz, M., The forest plus stars colouring problem, Communication at the Technical University of Ilmenau 1994.

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