# Vertex Disjoint Cycles in a Directed Graph

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**Abstract**

Let $D$ be a directed graph of order $n \geq 4$ and minimum degree at least $(3n-3)/2$. Let $n = n_1 + n_2$ where $n_1 \geq 2$ and $n_2 \geq 2$. Then $D$ contains two vertex-disjoint directed cycles of lengths $n_1$ and $n_2$ respectively. The result is sharp if $n \geq 6$: we give counter-examples if the condition on the minimum degree is relaxed.

## 1 Introduction

We discuss only finite simple graphs and strict digraphs and use standard terminology and notation from [3] except as indicated.

In 1963, Corrádi and Hajnal [4] investigated the maximum number of vertex-disjoint cycles in a graph. They proved that if $G$ is a graph of order at least $3k$ with minimum degree at least $2k$, then $G$ contains $k$ vertex-disjoint cycles. In particular, when the order of $G$ is exactly $3k$, then $G$ contains $k$ vertex-disjoint triangles. In 1984 El-Zahar [5] proved that if $G$ is a graph of order $n = n_1 + n_2$ with $n_i \geq 3$, $i = 1, 2$ and minimum degree at least $\lceil n_1/2 \rceil + \lceil n_2/2 \rceil$, then $G$ contains two vertex-disjoint cycles of lengths $n_1$, $n_2$, respectively. In 1991, Amar and Raspaud [1] investigated vertex-disjoint dicycles in a strongly connected digraph of order $n$ with $(n-1)(n-2)+3$ arcs. In this paper, we discuss two vertex-disjoint dicycles in a digraph, proving the following result and showing that it is sharp for all $n \geq 6$.

**THEOREM** *Let $D$ be a digraph of order $n \geq 4$ such that the minimum degree of $D$ is at least $(3n-3)/2$. Then $D$ contains two vertex-disjoint dicycles of lengths $n_1$ and $n_2$, respectively, for any integer partition $n = n_1 + n_2$ with $n_1 \geq 2$ and $n_2 \geq 2$.*

To prove our result, we recall some terminology and notation. Let $G$ be a graph and $D$ a digraph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set respectively, of $G$. We use $V(D)$ and $E(D)$ to denote the vertex set and arc set respectively, of $D$. A similar notation is used for the vertex sets and edge sets or arc sets of paths and cycles. The degree $d_G(x)$ or $d_D(x)$ of a vertex $x$ in $G$ or $D$ respectively is the number of edges or arcs incident on it. We use $\delta(G)$ and $\delta(D)$ for the minimum degree of a vertex in $G$ or $D$ respectively.

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For a vertex $u \in V(G)$ and a subgraph $H$ of $G$, we define $d_G(u, H)$ or $d(u, H)$ to be the number of vertices of $H$ that are adjacent to $u$ in $G$. For a vertex $x \in V(D)$ and a subdigraph $F$ of $D$ we define $d_D(x, F)$ similarly. If $F_1$ and $F_2$ are vertex-disjoint subdigraphs of $D$, then $e_D(F_1, F_2)$ denotes the number of arcs of $D$ joining a vertex of $V(F_1)$ to a vertex of $V(F_2)$. For a subset $U$ of $V(G)$, $G[U]$ is the subgraph of $G$ induced by $U$. Similarly, $D[X]$ is the subdigraph of $D$ induced by $X$ for any subset $X$ of $V(D)$. A graph or digraph is said to be *traceable* if it contains a Hamiltonian path or a Hamiltonian dipath, respectively.

For any integer $n$ we define $\epsilon_n$ to be 0 or 1 according to whether $n$ is even or odd. If $x$ and $y$ are vertices of $G$, we define $\epsilon(xy)$ to be 1 if $x$ and $y$ are adjacent, and 0 otherwise.

## 2 Proof of the Theorem

We begin with some elementary lemmas.

LEMMA 1 *Let $P$ be a path in a graph $G$. Let $z \in V(G) - V(P)$. If $d(z, P) \geq \frac{1}{2}|V(P)|$, then $G[V(P) \cup \{z\}]$ is traceable.*

**Proof:** The lemma is immediate, since $z$ must be adjacent to consecutive vertices of $P$ or to an end vertex of $P$. □

LEMMA 2 *Let $x$ and $y$ be the ends of a path $P$ of positive length in a graph $G$. If $d(x, P) + d(y, P) \geq |V(P)|$, then $G[V(P)]$ is Hamiltonian or isomorphic to $K_2$.*

**Proof:** See [6]. □

LEMMA 3 *Let $G_1$, $G_2$ be vertex-disjoint traceable induced subgraphs of a graph $G$, where $|G_1| = n_1$ and $|G_2| = n_2$, and suppose that $|E(G_1)| + |E(G_2)|$ is as large as possible subject to those conditions. Let $x$ and $y$ be vertices of $G_1$ and $G_2$ respectively. Let $H_1 = G_1 - x + y$ and $H_2 = G_2 - y + x$. If $H_1$ and $H_2$ are also traceable, then*

$$d(x, G_1) + d(y, G_2) \geq d(x, G_2) + d(y, G_1) - 2\epsilon(xy).$$

**Proof:** By hypothesis,

$$\begin{aligned} & |E(G_1)| + |E(G_2)| \geq |E(H_1)| + |E(H_2)| \\ = \; & |E(G_1)| + |E(G_2)| - d(x, G_1) - d(y, G_2) + d(x, G_2) + d(y, G_1) - 2\epsilon(xy), \end{aligned}$$

and the result follows. □

**Proof of the theorem:** Let $G$ be an undirected simple graph with $V(G) = V(D)$, where two distinct vertices $u$ and $v$ are adjacent if and only if $(u, v) \in E(D)$ and $(v, u) \in E(D)$. For any $x \in V(G)$ we have

$$\begin{aligned} d(x, G) &\geq 3(n-1)/2 - (n-1) \\ &= (n-1)/2, \end{aligned}$$

and so $\delta(G) \geq (n-1)/2$. Thus $G$ is traceable [2, p.135]. We may therefore choose two traceable induced subgraphs $G_1$ and $G_2$ such that $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$ and $|E(G_1)|+|E(G_2)|$ is as large as possible subject to these conditions. Let $P_1$ and $P_2$ be Hamiltonian paths of $G_1$ and $G_2$ respectively. Let $V(P_1) = \{x_1, x_2, \cdots, x_{n_1}\}$, where $x_i$ is adjacent in $P_1$ to $x_{i-1}$ for each $i > 1$. Similarly let $V(P_2) = \{y_1, y_2, \cdots, y_{n_2}\}$, where $y_j$ is adjacent in $P_2$ to $y_{j-1}$ for each $j > 1$.

**Case I:** Suppose that neither $G_1$ nor $G_2$ is Hamiltonian or isomorphic to $K_2$. Thus $d(x_1, G_1) + d(x_{n_1}, G_1) < n_1$ by Lemma 2, and so we may assume without loss of generality that $d(x_1, G_1) \leq (n_1 - 1)/2$. As this number must be an integer, we conclude that

$$d(x_1, G_1) \leq (n_1 - 2 + \epsilon_{n_1})/2. \tag{1}$$

Similarly we may assume that

$$d(y_1, G_2) \leq (n_2 - 2 + \epsilon_{n_2})/2. \tag{2}$$

Because $\delta(G) \geq \lceil (n-1)/2 \rceil = (n - \epsilon_n)/2$, it follows that

$$\begin{aligned} d(x_1, G_2) &\geq (n_1 + n_2 - \epsilon_n)/2 - (n_1 - 2 + \epsilon_{n_1})/2 \\ &= (n_2 + 2 - \epsilon_n - \epsilon_{n_1})/2 \\ &\geq (n_2 + \epsilon_{n_2})/2 \end{aligned} \tag{3}$$

since $\epsilon_n + \epsilon_{n_1} + \epsilon_{n_2} \leq 2$. (Note that $n$ is even if both $n_1$ and $n_2$ are odd.) Similarly

$$d(y_1, G_1) \geq (n_1 + \epsilon_{n_1})/2. \tag{4}$$

Let $L_1 = G_1 - x_1 + y_1$ and $L_2 = G_2 - y_1 + x_1$.

**Subcase A:** Suppose $L_1$ and $L_2$ are both traceable. By Lemma 3, together with (1) - (4) we have

$$(n_1 - 2 + \epsilon_{n_1})/2 + (n_2 - 2 + \epsilon_{n_2})/2 \geq (n_2 + \epsilon_{n_2})/2 + (n_1 + \epsilon_{n_1})/2 - 2\epsilon(x_1 y_1),$$

from which we infer that equality must hold in (1) - (4). In particular

$$(n_1 + \epsilon_{n_1})/2 = (n_1 + 2 - \epsilon_n - \epsilon_{n_2})/2,$$

and so $\epsilon_n + \epsilon_{n_1} + \epsilon_{n_2} = 2$. We deduce that

$$\begin{aligned} d_G(x_1) &= (n_1 - 2 + \epsilon_{n_1})/2 + (n_2 + \epsilon_{n_2})/2 \\ &= (n - 2 + \epsilon_{n_1} + \epsilon_{n_2})/2 \\ &= (n - \epsilon_n)/2. \end{aligned}$$

As $d_D(x_1) \geq \lceil (3n-3)/2 \rceil = (3n - 2 - \epsilon_n)/2$, it follows that $d_D(x_1) - d_G(x_1) \geq (3n-2-\epsilon_n)/2 - (n - \epsilon_n)/2 = n - 1$. Hence $x_1$ is adjacent in $D$ to every other vertex. A similar statement holds for $y_1$. A directed cycle in $D$ with vertex set $V(G_1)$ may

therefore be constructed by adjoining to $P_1$ an edge joining $x_1$ to $x_{n_1}$. Similarly $D$ has a directed cycle with vertex set $V(G_2)$, as required.

**Subcase B:** We may now suppose without loss of generality that $L_2$ is not traceable. Therefore $x_1$ cannot be adjacent to consecutive vertices or the end vertices of $P_2 - y_1$, and so $d(x_1, G_2) \leq (n_2 - \epsilon_{n_2})/2$. But $d(x_1, G_2) \geq (n_2 + \epsilon_{n_2})/2$ from (3). We conclude that $\epsilon_{n_2} = 0$, so that $n_2$ is even. Moreover $d(x_1, G_2) = n_2/2$, and from (1) and the inequality $\delta(G) \geq (n - \epsilon_n)/2$ it follows that $n$ and $n_1$ are odd and $d(x_1, G_1) = (n_1 - 1)/2$. Thus $d_G(x_1) = (n-1)/2$, and we find once again that $x_1$ is adjacent in $D$ to every other vertex. In particular, $x_1$ is adjacent in $D$ to $x_{n_1}$. If $y_1$ were adjacent in $D$ to $y_{n_2}$, then we would be done, and so we suppose that such is not the case.

Since $L_2$ is not traceable, $x_1$ is not adjacent to $y_{n_2}$. But $d(x_1, G_2) = n_2/2$ and $x_1$ is not adjacent to consecutive vertices of $P_2$. Therefore $x_1$ must be adjacent to $y_{2i+1}$ for each $i \geq 0$. It follows that $y_{n_2}$ is not adjacent to $y_{2i}$ for any $i \geq 1$, for otherwise $(P_2 - \{y_1y_2, y_{2i}y_{2i+1}\}) \cup \{x_1y_{2i+1}, y_{n_2}y_{2i}\}$ would be a Hamiltonian path in $L_2$. Since $y_{n_2}$ is also not adjacent to $y_1$, we infer that $d(y_{n_2}, G_2) \leq n_2 - n_2/2 - 1 = (n_2 - 2)/2$. In other words, (2) holds with $y_1$ replaced by $y_{n_2}$. We may therefore repeat the argument, with the rôles of $y_1$ and $y_{n_2}$ interchanged, in order to obtain the contradiction that $x_1$ is adjacent to $y_{n_2}$.

**Case II:** We may now assume without loss of generality that $G_1$ is Hamiltonian or isomorphic to $K_2$. We may also assume that $\delta(G_1) \geq (n_1 + \epsilon_{n_1})/2$, for if $d(x, G_1) \leq (n_1 - 2 + \epsilon_{n_1})/2$ for some $x \in V(G_1)$ then the argument of the previous case applies, since $x$ is an end of a Hamiltonian path of $G_1$.

The theorem clearly holds if $D[V(G_2)]$ is Hamiltonian. We therefore suppose it is not. As in the previous case we may assume that (2) holds. Hence (4) holds as before.

Define $H_1 = G_1 + y_1$ and $H_2 = G_2 - y_1$. There are two subcases.

**Subcase A:** Suppose there is no vertex $u \in V(H_1)$ such that $D[V(H_2) \cup \{u\}]$ is Hamiltonian. Then no vertex of $H_1$ is adjacent in $G$ to both $y_2$ and $y_{n_2}$.

**Subcase A (1):** Suppose $D[V(H_2)]$ is Hamiltonian. Let $V(H_2) = \{v_1, v_2, \cdots, v_{n_2-1}\}$ where $(v_{i-1}, v_i) \in E(D)$ for each $i > 1$ and $(v_{n_2-1}, v_1) \in E(D)$. For any $u \in V(H_1)$ let $I_u$ be the set of all $i$ such that $(v_i, u) \in E(D)$, and let $J_u$ be the set of all $j$ such that $(u, v_{j+1}) \in E(D)$, where $v_{n_2} = v_1$. Then $I_u \cap J_u = \phi$ since $D[V(H_2) \cup \{u\}]$ is not Hamiltonian. Therefore $d_D(u, H_2) = |I_u| + |J_u| = |I_u \cup J_u| \leq n_2 - 1$, and so

$$e_D(H_1, H_2) \leq (n_1 + 1)(n_2 - 1).$$

For each $u \in V(H_1)$ it follows that

$$d_D(u, H_1) \geq (3n - 2 - \epsilon_n)/2 - n_2 + 1.$$

Hence

$$2n_1 \geq n_1 + (n_1 + n_2 - \epsilon_n)/2,$$

so that

$$n_1 \geq n_2 - \epsilon_n. \tag{5}$$

On the other hand,

$$\begin{aligned}(n_1+1)(n_2-1) &\geq e_D(H_1,H_2)\\ &= \sum_{v\in V(H_2)} d_D(v) - 2|E(H_2)|\\ &\geq (3n-2-\epsilon_n)(n_2-1)/2 - 2(n_2-1)(n_2-2),\end{aligned}$$

and so

$$n_1 + 1 \geq (3n-2-\epsilon_n)/2 - 2(n_2-2).$$

Therefore

$$2n_1 + 2 \geq 3n_1 + 3n_2 - 2 - \epsilon_n - 4n_2 + 8,$$

so that

$$\begin{aligned}n_2 &\geq n_1 - \epsilon_n + 4\\ &\geq n_2 - 2\epsilon_n + 4\end{aligned}$$

from (5). We now have a contradiction.

**Subcase A (2):** Thus $D[V(H_2)]$ is not Hamiltonian. Consequently $d_G(y_2,H_2) + d_G(y_{n_2},H_2) \leq n_2 - 2$ by Lemma 2. Therefore

$$\begin{aligned}d_G(y_2) + d_G(y_{n_2}) &\leq n_1 + 1 + n_2 - 2\\ &= n-1,\end{aligned}$$

since no vertex of $H_1$ is adjacent to both $y_2$ and $y_{n_2}$. But

$$\begin{aligned}d_G(y_2) + d_G(y_{n_2}) &\geq 2(n-\epsilon_n)/2\\ &= n - \epsilon_n.\end{aligned}$$

Thus $\epsilon_n = 1$ and equality must hold above. Hence $d_G(y_2) = d_G(y_{n_2}) = (n-1)/2$. It follows that

$$\begin{aligned}d_D(y_2) - d_G(y_2) &\geq (3n-3)/2 - (n-1)/2\\ &= n-1,\end{aligned}$$

so that $y_2$ is adjacent in $D$ to every other vertex. Thus $y_2$ is adjacent to $y_{n_2}$, and we have the contradiction that $D[V(H_2)]$ is Hamiltonian.

**Subcase B:** Suppose there exists $u \in V(H_1)$ such that $D[V(H_2)\cup\{u\}]$ is Hamiltonian. Note that $u \neq y_1$ since $D[V(G_2)]$ is not Hamiltonian. Let $L = H_1 - u$. We may therefore assume that $L$ is not Hamiltonian or isomorphic to $K_2$, for otherwise we are done.

Since $\delta(G_1) \geq (n_1 + \epsilon_{n_1})/2$ we have $d(x, G_1) \geq (n_1 + \epsilon_{n_1})/2$ for each $x \in V(H_1) - \{y_1\}$. We suppose first that equality holds for some such $x \neq u$. In this case we shall show that $x$ is adjacent to $y_1$. Observe first that

$$\begin{aligned} d(x, G_2) &\geq (n - \epsilon_n)/2 - (n_1 + \epsilon_{n_1})/2 \\ &= (n_2 - \epsilon_n - \epsilon_{n_1})/2. \end{aligned}$$

Suppose $x$ is not adjacent to $y_1$. Since $G_1$ is Hamiltonian or isomorphic to $K_2$, $x$ is an end of a Hamiltonian path $P$ in $G_1$, and $d(y_1, G_1 - x) \geq (n_1 + \epsilon_{n_1})/2 > (n_1 - 1)/2$ by (4). Therefore $H_1 - x$ is traceable by Lemma 1.

**Subcase B (1):** Suppose $H_2 + x$ is also traceable. Then by Lemma 3, (2) and (4) we find that

$$(n_1 + \epsilon_{n_1})/2 + (n_2 - 2 + \epsilon_{n_2})/2 \geq (n_2 - \epsilon_n - \epsilon_{n_1})/2 + (n_1 + \epsilon_{n_1})/2.$$

In fact, equality must hold since $\epsilon_n + \epsilon_{n_1} + \epsilon_{n_2} \leq 2$. Therefore $d(y_1, G_1) = (n_1 + \epsilon_{n_1})/2$, and $d(y_1, G_2) = (n_2 - 2 + \epsilon_{n_2})/2$. Thus

$$\begin{aligned} d(y_1, G) &= (n_1 + \epsilon_{n_1})/2 + (n_2 - 2 + \epsilon_{n_2})/2 \\ &= (n - 2 + \epsilon_{n_1} + \epsilon_{n_2})/2, \end{aligned}$$

so that

$$\begin{aligned} d_D(y_1) - d_G(y_1) &\geq (3n - 2 - \epsilon_n)/2 - (n - 2 + \epsilon_{n_1} + \epsilon_{n_2})/2 \\ &= (2n - \epsilon_n - \epsilon_{n_1} - \epsilon_{n_2})/2 \\ &\geq n - 1. \end{aligned}$$

Again equality must hold. Moreover $y_1$ must be adjacent in $D$ to every other vertex. In particular, $y_1$ is adjacent to $y_{n_2}$, in contradiction to the fact that $D[V(G_2)]$ is not Hamiltonian.

**Subcase B (2):** Suppose $H_2 + x$ is not traceable. Then $x$ cannot be adjacent to consecutive vertices of $P_2$, or to $y_2$ or $y_{n_2}$. Therefore $d(x, H_2) < (n_2 - 1)/2$. But $d(x, G_2) \geq (n_2 - \epsilon_n - \epsilon_{n_1})/2 \geq (n_2 - 2)/2$ and $x$ is not adjacent to $y_1$. We are forced to the conclusion that $d(x, G_2) = (n_2 - 2)/2$, so that $n_2$ is even. Moreover $x$ is adjacent to $y_{2i+1}$ for each positive integer $i < n_2/2$. If $y_{n_2}$ is adjacent to $y_{2i}$ for some such $i$, then $(P_2 - \{y_1y_2, y_{2i}y_{2i+1}\}) \cup \{xy_{2i+1}, y_{n_2}y_{2i}\}$ is a Hamiltonian path in $H_2 + x$, contrary to hypothesis. Furthermore $y_{n_2}$ is not adjacent to $y_1$, and so $d(y_{n_2}, G_2) \leq (n_2 - 2)/2 = (n_2 - 2 + \epsilon_{n_2})/2$. Note that $G_2 - y_{n_2} + x$ is traceable since $x$ is adjacent to $y_{n_2-1}$. The argument of subcase B(1) then applies with $y_1$ and $y_{n_2}$ interchanged, yielding a contradiction.

We conclude that each $x \in V(L) - \{y_1\}$ satisfying $d(x, G_1) = (n_1 + \epsilon_{n_1})/2$ must be adjacent to $y_1$. For any $x \in V(L) - \{y_1\}$ it therefore follows that $d(x, H_1) \geq (n_1 + \epsilon_{n_1})/2 + 1$, and so $d(x, L) \geq (n_1 + \epsilon_{n_1})/2$. But $\delta(L) < n_1/2$ since $L$ is not Hamiltonian or isomorphic to $K_2$. Hence $d(y_1, L) \leq (n_1 - 2 + \epsilon_{n_1})/2$. On the other

hand, since $d(y_1, H_1) \geq (n_1 + \epsilon_{n_1})/2$ we deduce that $d(y_1, L) \geq (n_1 - 2 + \epsilon_{n_1})/2$. Therefore equality holds, and so $d(y_1, G_1) = (n_1 + \epsilon_{n_1})/2$. From (2) it follows that

$$\begin{aligned} d(y_1, G) &\leq (n - 2 + \epsilon_{n_1} + \epsilon_{n_2})/2 \\ &\leq (n - \epsilon_n)/2, \end{aligned}$$

so that $d_D(y_1) - d_G(y_1) \geq n - 1$. Thus $y_1$ is adjacent to $y_{n_2}$ in $D$, and again we have the contradiction that $D[V(G_2)]$ is Hamiltonian. □

To show that the condition in the theorem is sharp for each $n \geq 6$, we construct the following digraph $D_n$ of order $n$. For any positive integer $k$, define $K_k^*$ to be the complete digraph of order $k$, i.e., $K_k^*$ contains both $(u, v)$ and $(v, u)$ for any two distinct vertices $u$ and $v$ of $K_k^*$. The digraph $D_n$ consists of two vertex-disjoint complete subdigraphs $D'$ and $D''$ of order $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, respectively, and all the arcs $(u, v)$ with $u \in V(D')$ and $v \in V(D'')$. When $n$ is odd, $\delta(D) = (3n - 5)/2$. When $n$ is even, $\delta(D) = (3n - 4)/2$. Let $n = n_1 + n_2$ be any integer partition such that $n_1 \geq 2$, $n_2 \geq 2$ and $\{n_1, n_2\} \neq \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$. Then it is easy to see that $D_n$ does not contain two vertex-disjoint dicycles of lengths $n_1$ and $n_2$ respectively. It is our belief that if $D$ is strongly connected, then the condition can be improved. Note that the theorem does not hold if $n_1 = 1$ or $n_2 = 1$, even if loops are permitted. In this case $K_2^*$ gives a counterexample.

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