On the upper chromatic number of a hypergraph *

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Abstract. We introduce the notion of a co-edge of a hypergraph, which is a subset of vertices to be colored so that at least two vertices are of the same color. Hypergraphs with both edges and co-edges are called mixed hypergraphs. The maximal number of colors for which there exists a mixed hypergraph coloring using all the colors is called the upper chromatic number of a hypergraph H and is denoted by $\bar{\chi}(H)$. An algorithm for computing the number of colorings of a mixed hypergraph is proposed. The properties of the upper chromatic number and the colorings of some classes of hypergraphs are discussed. A greedy polynomial time algorithm for finding a lower bound for $\bar{\chi}(H)$ of a hypergraph H containing only co-edges is presented.

The cardinality of a maximum stable set of an all-vertex partial hypergraph generated by co-edges is called the co-stability number $\alpha_{\mathcal{A}}(H)$. A hypergraph H is called co-perfect if $\bar{\chi}(H') = \alpha_{\mathcal{A}}(H')$ for all its wholly-edge subhypergraphs H' . Two classes of minimal non co-perfect hypergraphs (the so called monostars and cycloids C^r_{2r-1}, $r \geq 3$) are found. It is proved that hypertrees are co-perfect if and only if they do not contain monostars as wholly-edge subhypergraphs.

It is conjectured that the $r-$uniform hypergraph H is co-perfect if and only if it contains neither monostars nor cycloids C^r_{2r-1}, $r \geq 3$, as wholly-edge subhypergraphs.

1. Introduction.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of sources of power supply such that the action time of any source is one quantum of time and all sources acting for any given quantum of time switch on and switch off synchronously.

Consider the following general constraints on their common work:

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1) let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$, $A_i \subseteq X$, $i = 1, \dots, k$, $k \geq 1$, be a family of subsets of X such that at least two sources from every A_i act for the same quantum of time;

2) let $\mathcal{E} = \{E_1, E_2, \dots, E_m\}$, $E_j \subseteq X$, $j = 1, \dots, m$, $m \geq 1$, be a family of subsets of X such that at least two sources from every E_j act for different quanta of time.

Call the set X with such constraints a *system* and denote it by $H = (X, \mathcal{A} \cup \mathcal{E})$. Suppose that system H is active ("working","alive") during any quantum of time if at least one source is active for this time.

We consider the following problem: *how can we schedule the system H in such a way that the time of working (which may be understood also as the life time of the whole system) is longest?*

In this paper we show that this problem may be formulated in terms of Hypergraph Coloring Theory and leads to the notion of the upper chromatic number of a hypergraph. We develop this Theory and show also that in many cases its methods can be successfully used for solving the problem.

In section 2 we consider two types of colorings (free and strict) and introduce the notion of a co-edge (equivalently called an anti-edge) of a hypergraph, which is a subset of vertices that is colored in a way that at least two vertices of the co-edge have the same color. Hypergraphs having co-edges only are called co-hypergraphs, and hypergraphs containing both edges and co-edges are called mixed hypergraphs. The upper chromatic number of a mixed hypergraph is the maximal number of colors for which there exists a coloring of the hypergraph H using all the colors.

In the problem above, if we denote the sources by vertices of a hypergraph and the given constraints by edges and co-edges, then in any hypergraph coloring every monochromatic subset of vertices represents a set of sources that may be switched on synchronously. Therefore, the initial scheduling problem is equivalent to the problem of finding the upper chromatic number and a corresponding coloring of a mixed hypergraph.

In section 3 a generalization of the connection-contraction algorithm (which we name the "splitting-contraction algorithm") for finding all colorings of a mixed hypergraph is developed. We prove that for mixed hypergraphs the following fundamental equality holds:

$$P(H, \lambda) = \sum_{i=\chi(H)}^{\bar{\chi}(H)} r_i(H) \lambda^{(i)},$$

where $P(H, \lambda)$ is the number of free colorings with λ colors, $r_i(H)$ is the number of strict colorings with i colors (the chromatic spectrum), χ, $\bar{\chi}$ are the lower and upper chromatic numbers (respectively) of a mixed hypergraph H and $\lambda^{(i)} = \lambda(\lambda - 1) \dots (\lambda - i + 1)$. The class of polynomials which may be chromatic for some mixed hypergraph is essentially larger than the class for ordinary hypergraphs, mainly because of interactions between edges and co-edges. For example, adding an edge to a hypergraph may decrease $\bar{\chi}$.

In section 4 we investigate the properties of $\bar{\chi}(H)$. It is shown that the so called co-bistars play an important role in problems concerning the upper chromatic number. We find that a mixed hypergraph H has $\bar{\chi}(H) = |X| - 1$ if and only if it represents a co-bistar.

It is well known that there are many interesting and important classes of bipartite hypergraphs [1,3], that is hypergraphs for which $\chi(H) = 2$. However the class of their opposites, that is the class of co-hypergraphs with $\bar{\chi}(H) = n - 2$, has appeared unexpectedly poor: it consists only of special unions of two co-bistars or of the so called holes.

We propose a consecutive greedy coloring algorithm of complexity $O(n^3k + n^2k^2)$ where n, k are the vertex and edge numbers respectively, for finding a lower bound for $\bar{\chi}$ and a corresponding coloring of any co-hypergraph.

Its main difference from the usual greedy hypergraph coloring algorithm is that at each step we can lose all but one used color; this circumstance prevents us from obtaining a direct estimate for the upper chromatic number. It is shown that the maximal number of colors that may be lost in the worst case at each coloring step does not exceed the value $O(H) + 1$, where

$$O(H) = \max_{Y \subseteq X} \min_{x \in Y} o(H/Y, x)$$

and $o(H/Y, x)$ is the difference between the degree and paired degree of a vertex x in a wholly-edge subhypergraph generated by the set $Y \subseteq X$. It appears that the class of co-hypergraphs having $O(H) = 0$ has the property that the greedy algorithm may be implemented without re-colorings of vertices. This class is large and, for example, includes the hypertrees. So, hypertrees represent the first known class of hypergraphs that play a special role in co-hypergraph colorings.

A method of monochromatic component re-coloring that is the opposite to the known method of bi-chromatic chain re-coloring in graph theory is proposed.

The cardinality of a maximum stable set in an all-vertex partial hypergraph generated by co-edges is called the co-stability number $\alpha_{\mathcal{A}}(H)$; we have that $\bar{\chi}(H) \leq \alpha_{\mathcal{A}}(H)$, and that the co-stability number plays a role for $\bar{\chi}(H)$ analogous to that which the maximum clique number $\omega(G)$ plays for the usual chromatic number $\chi(G)$ for a graph G. It is shown also how one can construct a hypergraph H with $\alpha_{\mathcal{A}}(H) - \bar{\chi}(H) > k$ for any $k \geq 0$ and $\bar{\chi} > 1$.

We introduce the notion of co-perfect hypergraph: a mixed hypergraph H is called co-perfect if $\bar{\chi}(H') = \alpha_{\mathcal{A}}(H')$ for all its wholly-edge subhypergraphs H'. Some classes of co-perfect and non co-perfect hypergraphs are discussed. Two classes of minimal non co-perfect hypergraphs are found: monostars, that have exactly one vertex common to all their edges, and cycloids $C^r_{2r-1}, r \geq 3$, that are isomorphic to the family of all paths on r vertices of a cycle C_{2r-1}. It is shown that hypertrees are co-perfect if and only if they do not contain monostars as wholly-edge subhypergraphs.

It is conjectured that the $r-$uniform co-hypergraph H is co-perfect if and only if it does not contain co-monostars and co-cycloids C^r_{2r-1} as wholly-edge subhypergraphs. This conjecture is analogous to the strong Berge's conjecture on perfect graphs.

2. Preliminaries

Let $X = \{x_1, x_2, \ldots, x_n\}$, $n \geq 1$, be a finite set, $\mathbf{S} = \{S_1, S_2, \ldots, S_t\}$, $t \geq 1$, be a family of subsets of X.

The pair $H = (X, \mathbf{S})$ is called a hypergraph on X (cf. [1,3]). For any subfamily $F \subseteq \mathbf{S}$ we call the hypergraph $H_F = (X, F)$ the *all-vertex partial hypergraph*, generated by the family F.

For any subset $Y \subseteq X$ we call the hypergraph $H/Y = (Y, \mathbf{S}')$ a *wholly-edge subhypergraph* of the hypergraph H if $\mathbf{S}'$ consists of all those subsets in $\mathbf{S}$ that are completely contained in Y.

Let the hypergraph $H = (X, \mathbf{S})$ be given, with $|X| = n$ and $\mathbf{S} = \mathcal{A} \cup \mathcal{E}$, where both $\mathcal{A}$ and $\mathcal{E}$ are subfamilies of $\mathbf{S}$, in particular, $\mathcal{E}$ and/or $\mathcal{A}$ may be empty. Let $\mathcal{A} = \{A_1, \ldots, A_k\}$, $I = \{1, \ldots, k\}$, $\mathcal{E} = \{E_1, \ldots, E_m\}$, $J = \{1, \ldots, m\}$, and $|A_i| \geq 2$, $i \in I$, and $|E_j| \geq 2$, $j \in J$.

We call every E_j, $j \in J$, an *"edge"*, and every A_i, $i \in I$, a *"co-edge"* or, equivalently, an *"anti-edge"*. We use the prefix "co-" when a statement concerns the sets from $\mathcal{A}$. In particular, if $\mathcal{E} = \emptyset$, then $H = H_{\mathcal{A}}$ will be called a *"co-hypergraph"*. In order to emphasize that for the general hypergraph H it may be that $\mathcal{A} \neq \emptyset$ and/or $\mathcal{E} \neq \emptyset$ we call H a *mixed hypergraph* [7]. Other terminology that is not explained here may be found in [1,3]. Let us have $\lambda \geq 0$ colors.

Definition *2.1 [7] A free coloring of a mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ with λ colors is a coloring of its vertices X in such a way that the following four conditions hold:*

1) any co-edge A_i, $i \in I$, has at least two vertices of the same color;

2) any edge E_j, $j \in J$, has at least two vertices colored differently;

3) the number of colors used is not greater than λ;

4) all the vertices are colored.

Note that this definition of a coloring generalizes all those contained in [3] that correspond to the case $\mathcal{A} = \emptyset$. Now we can say that the edges represent non-monochromatic subsets, and the co-edges represent the not all differently colored subsets of the vertices. Condition 1) of Definition 2.1 justifies the prefix 'co-'(equivalently 'anti-') because it means indeed that in any further considerations in colorings at least two vertices of any co-edge are considered as one vertex; in particular, a co-edge of cardinality 2 is equivalent to a single vertex. Thus from this view-point (only) a co-edge is not a usual set. Colorings in which not all the vertices of a hypergraph need be colored will be investigated in a separate paper.

Two free colorings of a hypergraph H are said to be different, if there exists at least one vertex that changes color when passing from one coloring to the other. Let $P(H, \lambda)$, $\lambda \geq 0$, be the chromatic polynomial of a hypergraph H, which is the number of different free colorings of H with λ colors [cf. 1].

Definition *2.2.[7] A free coloring of a hypergraph H with $i \geq 0$ colors is said to be a strict coloring, if exactly i colors are used.*

So, strict colorings exist only for i such that $1 \leq i \leq n$. Let us say that two strict colorings of H are *different* if there exist two vertices of H that have the same color for one of these colorings and different colors for the other (cf. [2]).

Definition 2.3. *[7] The maximal i for which there exists a strict coloring of a mixed hypergraph H with i colors is called the upper chromatic number of H and is denoted by $\bar{\chi}(H)$.*

Let $r_i(H)$ be the number of strict colorings of a hypergraph H with $i \geq 1$ colors (cf. [2]). Let $\chi(H)$ be the usual (we shall call it sometimes the lower) chromatic number of H [1,3]. We associate with the hypergraph H the vector $R(H) = (r_1, r_2, \ldots, r_n) \in \mathbf{R}^n$ and call it the *chromatic spectrum* of H; hence $R(H) = (0, \ldots, 0, r_\chi, \ldots, r_{\bar{\chi}}, 0, \ldots 0)$.

Definition 2.4. *[7] A mixed hypergraph H in which at least one pair of vertices cannot be colored because of constraints collision is called uncolorable; for such a hypergraph we put $\chi(H) = \bar{\chi}(H) = 0$.*

Definition 2.5. *[7] The value $\chi_m(H) = (\chi(H) + \bar{\chi}(H))/2$ is called the middle chromatic number of a hypergraph H.*

Definition 2.6. *[7] For colorable mixed hypergraphs the value $b(H) = \bar{\chi} - \chi + 1$ is called the breadth of the chromatic spectrum of H.*

If $\chi_m(H)$ is not an integer, then it means that $b(H)$ is even. If $\mathcal{A} = \emptyset$, then $\bar{\chi}(H) = n$, and we have thus a usual hypergraph coloring. If $\mathcal{E} = \emptyset$, then $\chi(H) = 1$, and we have unusual colorings.

Moreover, if $\mathcal{A} \neq \emptyset$ and $\mathcal{E} \neq \emptyset$, then one can easily construct, for every $n \geq 2$, an uncolorable hypergraph H (for which we defined $\chi(H) = \bar{\chi}(H) = 0$); this is possible only for mixed hypergraphs.

For example, any complete graph K_n, $n \geq 2$, with at least one added co-edge with cardinality ≥ 2 cannot be colored.

Evidently, any mixed hypergraph with $\bar{\chi}(H_{\mathcal{A}}) < \chi(H_{\mathcal{E}})$ is uncolorable.

3. Colorings.

Now in order to calculate $P(H, \lambda)$ and $R(H)$ for any mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ we provide the following 5 rules.

1) Let X' be a subset of X. If every pair of vertices of X' is an edge of H, and X' itself is a co-edge, then H is uncolorable. Similarly, if X' is a vertex set of a connected subgraph consisting of co-edges of cardinality 2, and X' itself is an edge, then H also is uncolorable (**elimination**).

2) If $E_i \subseteq E_j$, $i, j \in J$, then $P(H, \lambda) = P(H - E_j, \lambda)$, $R(H) = R(H - E_j)$ (**clearing**).

3) If $A_i \subseteq A_j$, $i, j \in I$, then $P(H, \lambda) = P(H - A_j, \lambda)$, $R(H) = R(H - A_j)$ (**co-clearing**).

4) If $A_t = \{x_k, x_l\}$, for some $t \in I$ and $x_k, x_l \in X$, such that $A_t \neq E_s$ for any $s \in J$, then

$$P(H,\lambda) = P(H_1,\lambda), \quad R(H) = R(H_1), \text{ where}$$
$$H_1 = (X_1, \mathcal{A}^1 \cup \mathcal{E}^1),\ X_1 = (X \backslash \{x_k, x_l\}) \cup \{y\}, \quad y \text{ is a new vertex;}$$

if $x_k \in E_j$, or $x_l \in E_j$, $j \in J$, then $E_j^1 = (E_j \backslash \{x_k, x_l\}) \cup \{y\}$, otherwise $E_j^1 = E_j$;

if $x_k \in A_i$, or $x_l \in A_i$, $i \in I$, then $A_i^1 = (A_i \backslash \{x_k, x_l\}) \cup \{y\}$, otherwise $A_i^1 = A_i$, (**contraction**).

5) If $\{x_k, x_l\} \notin \mathcal{E}$ and $\{x_k, x_l\} \notin \mathcal{A}$, then

$$P(H,\lambda) = P(H_1,\lambda) + P(H_2,\lambda), \quad R(H) = R(H_1) + R(H_2),$$

where

$$H_1 = (X, \mathcal{A} \cup \mathcal{E}_1),\ \mathcal{E}_1 = \mathcal{E} \cup \{x_k, x_l\},$$
$$H_2 = (X, \mathcal{A}_1 \cup \mathcal{E}),\ \mathcal{A}_1 = \mathcal{A} \cup \{x_k, x_l\} \text{ (\textbf{splitting})}.$$

The algorithm that allows us to compute $P(H,\lambda)$ and $R(H)$ for any mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ is a generalization of the connection-contraction algorithm [2,4]. The idea is to find a pair of vertices that does not belong to the edge and co-edge sets, and to split all colorings of H into two classes with respect to this pair. Further, by implementing elimination, clearing, co-clearing and contraction (in that order) the initial problem is recurrently reduced to the same one for the new pair of "simpler" hypergraphs (in sense that one of them has fewer vertices and another has more edges of cardinality 2).

Finally, we obtain a list of complete graphs. We call this algorithm the "splitting-contraction algorithm" and present it in the following form:

ALGORITHM 1 (splitting-contraction).

INPUT: an arbitrary mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$;
OUTPUT: a list Z of complete graphs;

STEP 0. Add the hypergraph H to the empty list Y, and set $Z = \{\emptyset\}$.

STEP 1. Verify the condition of elimination for each hypergraph from Y; delete uncolorable hypergraphs from Y.

STEP 2. Implement clearing, co-clearing, and after that contraction for all hypergraphs from Y.

STEP 3. Implement one splitting in each hypergraph from Y, where possible; delete complete graphs from the list Y and include them in the list Z; if splitting is implemented for at least one hypergraph, then go to STEP 1, else go to STEP 4.

STEP 4. OUTPUT: list $Z = \{K_{n_1}, K_{n_2}, \ldots, K_{n_t}\}$ of complete graphs. End.

Although Algorithm 1 is exponential, it is possible to find polynomial and effective modifications for some classes of good hypergraphs.

Theorem *3.1. For any mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$, Algorithm 1 finds the chromatic polynomial $P(H, \lambda)$ and the chromatic spectrum $R(H)$, and the following equality holds:*

$$P(H, \lambda) = \sum_{i=\chi(H)}^{\bar{\chi}(H)} r_i(H)\lambda^{(i)}.$$

Proof. If α_i is the number of complete graphs with i vertices in the list Z, then it follows from rules 1) - 5) and Algorithm 1 that

$$P(H, \lambda) = \sum_{i=1}^{n} \alpha_i P(K_i, \lambda).$$

Since rules 1) - 5) and the whole algorithm are equivalent for $P(H, \lambda)$ as well as for $R(H)$, we have also

$$r_j(H) = \sum_{i=1}^{n} \alpha_i r_j(K_i), \; j = 1, \ldots n.$$

Hence from

$$r_j(K_i) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$$

we conclude that $\alpha_i = r_i$, $i = 1, \ldots, n$. Thus the theorem follows from $P(K_i, \lambda) = \lambda^{(i)} = \lambda(\lambda - 1) \ldots (\lambda - i + 1)$. □

Theorem 1 shows that the chromatic spectrum $R(H)$ uniquely determines the chromatic polynomial $P(H, \lambda)$, and vice versa. No simple criterion is known for an arbitrary polynomial to be the chromatic polynomial of a graph or a hypergraph. However, the class of polynomials that may be chromatic for mixed hypergraphs is essentially larger than the class for ordinary hypergraphs because of interactions between edges and co-edges. As we shall see, such interactions are not simple and bring many new properties to hypergraph colorings.

For example, for $H = (X, \mathcal{A} \cup \mathcal{E})$, where $X = \{1, 2, 3\}$, $\mathcal{A} = A_1 = \{1, 2, 3\}$, $\mathcal{E} = E_1 = \{1, 2, 3\}$, we have $Z = \{K_2, K_2, K_2\} = \{3K_2\}$, $P(H, \lambda) = 3\lambda^{(2)} = 3\lambda^2 - 3\lambda$, $R(H) = (0, 3, 0)$, $\chi = \bar{\chi} = \chi_m = 2$, $b(H) = 1$ and the corresponding three colorings are the following: $(\alpha\alpha\beta)$, $(\alpha\beta\alpha)$ and $(\beta\alpha\alpha)$.

Consider another example. Let $H = (X, \mathcal{A} \cup \mathcal{E})$, where $X = \{1, 2, 3, 4, 5\}$, $\mathcal{A} = \{(1, 2, 3), (1, 3, 4), (1, 4, 5), (1, 5, 2)\}$, $\mathcal{E} = \{(3, 5)\}$; we have that $\bar{\chi}(H) = 3$, and after adding the edge (2,4) we obtain the new hypergraph H_1, for which $\bar{\chi}(H_1) = 2$. It is not hard to see, in general, that adding one co-edge to a hypergraph H can increase $\chi(H)$; and adding one edge to a mixed hypergraph can decrease $\bar{\chi}(H)$.

Let $H = (X, \mathcal{A} \cup \mathcal{E})$ be a mixed hypergraph, $H_{\mathcal{E}}$ the all-vertex partial hypergraph obtained from the edge set $\mathcal{E}$, and $H_{\mathcal{A}}$ the all-vertex partial hypergraph obtained from the co-edge set $\mathcal{A}$. Then the following inequalities hold:

$$\chi(H_{\mathcal{E}}) \leq \chi(H) \leq \bar{\chi}(H) \leq \bar{\chi}(H_{\mathcal{A}}).$$

There is one more unusual property of co-hypergraph colorings, which is impossible for hypergraphs with $\mathcal{A} = \emptyset$.

We say that a co-edge A_i of the mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ is *dead* if A_i does not contain any other co-edge, and $R(H) = R(H - A_i)$. One can see, for instance, that any co-edge of the co-hypergraph $H = (X, \mathcal{A})$, where $X = (1, 2, 3, 4)$ and $\mathcal{A} = \{(1,2,3), (1,3,4), (1,2,4), (2,3,4)\}$, is dead, because $R(H) = R(H - A_j) = (1, 7, 0, 0)$, $j = 1, 2, 3, 4$.

4. Upper chromatic number and co-perfect hypergraphs.

Let us discuss several properties of the upper chromatic number of a hypergraph and related values.

Definition 4.1. *A set $T \subseteq X$ is called a bitransversal of a mixed hypergraph $H = (X, \mathcal{E} \cup \mathcal{A})$ if $|T \cap E_i| \geq 2$ for any $i \in J$. The cardinality of a minimum bitransversal is denoted by $\tau_2(H_{\mathcal{E}})$. If $\mathcal{E}$ does not contain any edge of cardinality ≥ 2, then we put $\tau_2(H_{\mathcal{E}}) = 1$.*

A co-bitransversal of a mixed hypergraph and $\tau_2(H_A)$ are defined similarly if we replace $\mathcal{E}$ by $\mathcal{A}$ in the above definition.

Theorem 4.2. *If for a mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ there exist a minimum bitransversal $T_{\mathcal{E}} \subseteq X$ of the hypergraph $H_{\mathcal{E}}$ and a minimum co-bitransversal $T_{\mathcal{A}} \subseteq X$ of the co-hypergraph $H_{\mathcal{A}}$ such that $|T_{\mathcal{E}} \cap T_{\mathcal{A}}| \leq 1$, then the breadth of the chromatic spectrum of H satisfies the inequality:*

$$b(H) \geq |X| - \tau_2(H_{\mathcal{E}}) - \tau_2(H_{\mathcal{A}}) + 2.$$

Proof. Color all the vertices of $T_{\mathcal{E}}$ with different colors. After that color the vertices of $T_{\mathcal{A}}$ with the color of the common vertex (if $T_{\mathcal{E}} \cap T_{\mathcal{A}} = \emptyset$, then color $T_{\mathcal{A}}$ with the first color). Color the vertices of $X - \{T_{\mathcal{E}} \cup T_{\mathcal{A}}\}$ with the first color. Then we obtain a coloring of H. Hence $\chi(H) \leq \tau_2(H_{\mathcal{E}})$. If we color all the vertices of $X - \{T_{\mathcal{E}} \cup T_{\mathcal{A}}\}$ differently with the colors $|T_{\mathcal{E}}| + 1, |T_{\mathcal{E}}| + 2, \ldots$, then we again obtain a coloring of H. Therefore $\bar{\chi}(H) \geq |X| - \tau_2(H_{\mathcal{A}}) + 1$. Since $b(H) = \bar{\chi}(H) - \chi(H) + 1$, the theorem follows.□

Definition 4.3. *A mixed hypergraph is called a co-bistar if there exists a co-bitransversal of cardinality 2, which is not an edge.*

Definition 4.4. *A mixed hypergraph is called a hole if there exists a minimum co-bitransversal of cardinality three, which is not an edge, and any two vertices of which are not an edge.*

So, for a co-bistar H, $\tau_2(H_{\mathcal{A}}) = 2$, and for a hole H, $\tau_2(H_{\mathcal{A}}) = 3$.

Note that $\bar{\chi}(H) = |X|$ if and only if $\mathcal{A} = \emptyset$.

Theorem 4.5. *For any mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$, the following conditions are equivalent:*

1) $\bar{\chi}(H) = |X| - 1$;

2) H is a co-bistar.

Proof. If $\bar{\chi}(H) = n - 1$, then there are only two vertices, say x_1, x_2, which have the same color in some strict coloring of H with $\bar{\chi}$ colors. If at least one of x_i, $i = 1, 2$, does not belong to some co-edge A_i, $i \in I$, then all the vertices in A_i are colored differently, in contradiction to the definition of a co-edge. Hence H is a co-bistar.

Conversely, color the co-bitransversal x_1, x_2 with the first color and then the rest of the vertices all with different colors; this gives a strict coloring of H using $n - 1$ colors. Hence $\bar{\chi} \geq n - 1$. Since $\mathcal{A} \neq \emptyset$, $\bar{\chi}(H) \leq n - 1$, and the theorem follows.□

Theorem 4.6. *For any mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$, the following conditions are equivalent:*

1) $\bar{\chi}(H) = |X| - 2$;

2) H is not a co-bistar, and H is either a union of two co-bistars without intersecting co-bitransversals, or a hole.

Proof. 1) $\Rightarrow$ 2). Let $\bar{\chi}(H) = n - 2$. Consider any strict coloring of H with $\bar{\chi}(H)$ colors. There are two possibilities:

a) There are two vertices, say x_1, x_2, colored with the first color, and two other vertices, say x_3, x_4, colored with the second color and the rest of the vertices are all colored with different colors; since any co-edge must have two vertices colored with the same color, and H is not a co-bistar (because if it was, due to Theorem 4.5 $\bar{\chi}(H)$ would equal $|X| - 1$), it follows that H is a union of two co-bistars without intersecting co-bitransversals.

b) Some three vertices have the same color; hence every co-edge contains at least two of them. Since by Theorem 4.5 H is not a co-bistar, it follows that these three vertices compose a minimal co-bitransversal; therefore H is a hole.

2) $\Rightarrow$ 1) is obvious. □

Let $H = (X, \mathcal{A})$ be a co-hypergraph, and let $\mathcal{A}(x)$ denotes the set of co-edges containing vertex $x \in X$. We say that vertices x and y are adjacent if $\mathcal{A}(x) \bigcap \mathcal{A}(y) \neq \emptyset$.

Call the set $\mathcal{A}(x) \cap \mathcal{A}(y)$ a **co-bistar of the vertex** $x \in X$ with respect to the vertex y and denote it by $BS(x, y)$. Evidently, $BS(x, y) = BS(y, x)$.

So, every vertex y that is adjacent to x forms some co-bistar $BS(x, y)$. Some co-bistars of a given vertex may coincide.

Call the value

$$p(x) = \max\{|BS(x, y)| : y \text{ is adjacent to } x\}$$

the **paired degree** of a vertex x.

Call the value

$$o(H, x) = |\mathcal{A}(x)| - p(x)$$

the **originality of a vertex** x in the co-hypergraph H.

Hence, $o(H, x) \geq 0$, and $o(H, x) = 0$ means that there exists some other vertex $y \in X$ which is contained in at least the same set of co-edges. If the vertices of a hypergraph represent different objects in real life, and the edges are the subsets having some properties (each property is an edge), then an object x with $o(H, x) = 0$ is "not original" because there exists at least one other object with at least the same properties. So, the originality of a vertex is a measure of "similarity with its neighbors".

Definition *4.7. The value*

$$O(H) = \max_{Y \subseteq X} \min_{x \in Y} o(H/Y, x)$$

is called the originality of a hypergraph H.

Let the co-hypergraph H be colored. Denote the color of vertex x by $color(x)$.

Definition *4.8. The set $M \subseteq X$ of vertices is called a monochromatic component of vertex x, and is denoted by $MC(x)$, if the following conditions hold:*

1) $x \in M$;

2) for any $y \in M$, $color(y) = color(x)$;

3) for every $y \in M$ there exists a co-chain, say $(x, A_1, x_1, A_2, x_2, \ldots, x_{t-1}, A_t, y)$ connecting x and y and such that all the vertices $x_1, x_2, \ldots, x_{t-1}$ belong to M.

4) M is maximal with respect to inclusion.

Now in order to find a lower bound for the upper chromatic number we propose a greedy coloring algorithm for an arbitrary co-hypergraph $H = (X, \mathcal{A})$. The idea is to find some good ordering of the vertices and greedily color H successively, maximally using the local information. Namely, at each step we use a new color for the next vertex and verify the correctness of the coloring obtained; if the coloring is wrong, then we re-color some monochromatic component meeting at the neighborhood of the given vertex in order to guarantee the correctness of the new coloring and minimize the losses of used colors. This method of re-coloring monochromatic components is the opposite of the known method of re-coloring bi-chromatic chains from graph theory.

The principal difference from the greedy hypergraph coloring algorithm is that at each step we can lose all but one used color; this circumstance prevents us from finding an exact estimate of the upper chromatic number.

ALGORITHM 2 (greedy co-hypergraph coloring).

INPUT: An arbitrary co-hypergraph $H = (X, \mathcal{A})$, $|X| = n$.

OUTPUT: A strict coloring of H in some number of colors .

STEP 0. Set the list of used colors $U := \{\emptyset\}$.

STEP 1. Set $i := n$; declare $H_n := H$; find a vertex of minimum originality and label it x_n.

STEP 2. Set $i := i - 1$; if $i = 0$, then go to STEP 5.

STEP 3. Form a wholly-edge co-hypergraph $H/\{X - \{x_n, \ldots, x_{i+1}\}\} = H_i$.

STEP 4. Find a vertex of minimum originality in H_i and label it x_i; go to STEP 2.

STEP 5. Color the vertex x_1 with the first color; set $U := U \cup \{1\}$; $i := 1$, $new := 2$.

STEP 6. Set $i := i + 1$; if $i = n$, then go to STEP 10; color the vertex x_i of H_i with the *new* color; set $U := U \cup \{\text{new}\}$, $new := new + 1$.

STEP 7. Verify the correctness of the coloring of H_i; if there are no co-edges with all their vertices of different colors, then go to STEP 6.

STEP 8. Choose in H_i a neighbor y of the vertex x_i, that is contained in a largest set of all differently colored co-edges. If $color(x_i) = new$, then re-color x_i with $color(y)$, set $U := U - \{new\}$, $new := new - 1$, and go to STEP 7.

STEP 9. Re-color all the vertices from a monochromatic component $MC(y)$ with the $color(x_i)$ and go to STEP 7.

STEP 10. Renumber the colors of the list U in increasing order; end.

Complexity. Let us suppose that $H = (X, \mathcal{A})$ is represented by its incidence matrix, $|X| = n$, $|\mathcal{A}| = k$. Since finding the originality of a vertex requires $O(nk)$ steps, then finding the minimum originality requires $O(n^2k)$ steps. Hence STEPs 1-4 may be implemented in the worst case in $O(n^3k)$ steps.

Verifying the correctness of the coloring at STEP 7 requires $O(nk^2)$ steps. Choosing the vertex y, finding and re-coloring a monochromatic component $MC(y)$ at STEPs 8-9 takes at most $O(n^2k)$ steps. Hence the second part of the algorithm requires at most $O(n^2k + nk^2)$ steps. Consequently, the whole algorithm may be implemented in $O(n^3k + nk^2)$ steps. There are possibilities to improve this bound by using special data structures and techniques.□

EXAMPLE.

Consider the co-hypergraph $H = (X, \mathcal{A})$ such that $X = (1,2,3,4,5)$, $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\}$, $A_1 = (1,2,3)$, $A_2 = (2,3,4)$, $A_3 = (3,4,5)$, $A_4 = (4,5,1)$, $A_5 = (5,1,2)$. Declare $H_5 = H$. Since all the vertices have the same originality 1, let us start with the first vertex: $x_5 = 1$.

Form the wholly-edge co-hypergraph $H_4 = (X_4, \mathcal{A}_4)$ with $X_4 = (2,3,4,5)$, $\mathcal{A}_4 = \{A_2, A_3\}$. The first vertex with minimal originality is $x_4 = 2$.

Form the wholly-edge co-hypergraph $H_3 = (X_3, \mathcal{A}_3)$ with $X_3 = (3,4,5)$, $\mathcal{A}_3 = \{A_3\}$. The first vertex with minimal originality is $x_3 = 3$.

Form the wholly-edge co-hypergraph $H_2 = (X_2, \mathcal{A}_2)$ with $X_2 = (4,5)$, $\mathcal{A}_2 = \{\emptyset\}$. The first vertex with minimal originality is $x_2 = 4$.

Form the wholly-edge co-hypergraph $H_1 = (X_1, \mathcal{A}_1)$ with $X_1 = (5)$, $\mathcal{A}_1 = \{\emptyset\}$. The unique and last vertex is $x_1 = 5$.

These were the results of steps 1-4. Now start coloring. Let $c_i = color(x_i)$, and denote the coloring of the respective hypergraph by the vector $C = (c_1, c_2, c_3, c_4, c_5)$.

STEP 5. $C = (0,0,0,0,1)$. STEP 6. $C = (0,0,0,2,1)$. STEP 7. There are no all polychromatic co-edges in co-hypergraph H_2.

STEP 6. $C = (0,0,3,2,1)$. STEP 7. Anti-edge A_3 is all polychromatic.

STEP 8. Re-coloring: $C = (0,0,2,2,1)$.

STEP 7. There are no all polychromatic co-edges in the co-hypergraph H_3.

STEP 6. $C = (0,3,2,2,1)$.

STEP 7. There are no all polychromatic co-edges in the co-hypergraph H_4.

STEP 6. $C = (4, 3, 2, 2, 1)$.

STEP 7. The co-edges A_1, A_4, A_5 are each all polychromatic in the co-hypergraph H_5.

STEP 8. Vertices 2 and 5 in the hypergraph H_5 are contained in the largest number 2 of co-edges with all their vertices of different colors. Choose vertex 5. Re-coloring: $C = (1, 3, 2, 2, 1)$.

STEP 7. The co-edge A_1 is still all polychromatic.

STEP 8. Vertices 2 and 3 in the hyperegraph H_5 are contained in one co-edge with all its vertices of different colors. Choose vertex 3.

STEP 9. Re-coloring the monochromatic component $MC(3) : C = (1, 3, 1, 1, 1)$.

STEP 7. $H_1 = H$ is colored correctly.

STEP 10. Renumber the colors in increasing order: $C = (1, 2, 1, 1, 1)$.

End.

Remark. In contrast to the usual hypergraphs and co-hypergraphs the problem of finding at least one coloring for a mixed hypergraph in the general case is NP-complete. This follows from the fact that in the special case when $\chi(H) = \bar{\chi}(H)$ the problem is equivalent to finding both chromatic numbers. Mixed hypergraphs with $\chi(H) = \bar{\chi}(H)$ are still general enough. To see this, consider an arbitrary hypergraph $H = (X, \mathcal{E})$, with $|X| = n$, $\chi(H) \neq n$, and construct the sequence of mixed hypergraphs

$$H^r = (X, \mathcal{K}_n^r \cup \mathcal{E}),$$

where $\mathcal{K}_n^r$ is the r-uniform complete co-hypergraph [3, p.5], consisting of all the r-subsets of X, $r = n, n-1, \ldots, 2$. Hence

$$\bar{\chi}(H_A^r) = \bar{\chi}(X, \mathcal{K}_n^r \cup \emptyset) = \bar{\chi}(\mathcal{K}_n^r) = r - 1,$$

because we cannot use more than $r-1$ colors. Consider the inequalities

$$\chi(H_{\mathcal{E}}^r) \leq \chi(H^r) \leq \bar{\chi}(H^r) \leq \bar{\chi}(H_A^r) = r - 1$$

where $\chi(H_{\mathcal{E}}^r) = \chi(H)$ has a fixed value. There exists exactly one r such that equalities hold throughout above.

Theorem 4.9. The greedy co-hypergraph coloring algorithm finds the originality $O(H)$ for any co-hypergraph H.

Proof. Let t be the maximal value of the minimal originalities over all vertices in the order generated by STEPS 1-4. It is clear that $t \leq O(H)$.

Suppose that $t \leq O(H) - 1$. Hence in some wholly-edge subhypergraph $H^* \subseteq H$ there exists a vertex y such that

$$o(H^*, y) = \min_z \ o(H^*, z) = O(H) \geq t + 1.$$

It is easy to see that the originality of any vertex is a monotone function relative to wholly-edge subhypergraph inclusion. This implies that the first vertex of H^* that was deleted by the algorithm had originality $\geq t + 1$, and this contradicts the definition of t. Consequently, $t = O(H)$.□

The co-hypergraph $H = (X, \mathcal{A})$ is called a *co-hypertree* (arboreal [3, p.186]), if there exists a tree T such that every co-edge of H is the vertex set of a subtree of T.

Theorem 4.10. *If $H = (X, \mathcal{A})$ is a co-hypertree, then $O(H) = 0$.*

Proof. Induction on $|X| = n$. For $n = 1, 2$ the assertion is trivial. Assume it is true for any hypertree with $< n$ vertices. Consider a vertex x that is terminal in the corresponding tree T. Since any co-edge of H has cardinality at least 2, $o(H, x) = 0$. From this and $O(H/Y) = 0$ for $Y \subset X$ (by the induction hypothesis since H/Y is a co-hypertree) it follows that $O(H) = 0$.□

Hence, the class of co-hypertrees are the first known class of co-hypergraphs that occupies a special place in co-hypergraph colorings. In general, the class of co-hypergraphs having $O(H) = 0$ is much larger.

Theorem 4.11. *The number of colors that may be lost at STEPS 7-9 of Algorithm 2 does not exceed the value $O(H) + 1$.*

Proof. Let us suppose that we have the worst case at step 7, i.e. all co-edges containing vertex x_i in H_i have all their vertices of different colors. Remember that in H_i for the vertex x_i, there exists a neighbor y forming the largest co-bistar of x_i. If we re-color vertex x_i with $color(y)$, then we are losing one color (new) and at the same time correctly color $p(x_i)$ co-edges. Hence, in the worst case, there remain at most $o(H_i, x_i) = |\mathcal{A}(x_i)| - p(x_i)$ co-edges that still are colored with all different colors. Therefore, when re-coloring one vertex together with its monochromatic component from each of these co-edges with $color(x_i)$, we are losing again at most $o(H_i, x_i)$ colors. Consequently, the total number of colors lost is not greater than $o(H_i, x_i) + 1$.

Since for any i, $1 \leq i \leq n$

$$o(H_i, x_i) \leq O(H) = \max_{Y \subseteq X} \min_{x \in Y} o(H/Y, x)$$

the theorem follows.□

Theorem 4.12. If $|U| = p$ is the number of colors used by Algorithm 2, then $\bar{\chi}(H) \geq p$.

Proof. This follows immediately from Algorithm 2.

Definition 4.13. *For the mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$, a set $P \subseteq X$ is said to be co-stable if it does not contain any co-edge A_i, $i \in I$. The co-stability number $\alpha_{\mathcal{A}}(H)$ is the cardinality of maximum co-stable set in H.*

It follows from this definition that a maximum co-stable set is the largest set that could possibly be colored with all different colors. Although $\alpha_{\mathcal{A}}(H)$ equals the usual stability number of an all-vertex partial hypergraph $H_{\mathcal{A}} = (X, \mathcal{A})$, in mixed hypergraphs it plays a completely different role.

For the mixed hypergraph H, denote by $m_i(H)$ ($\bar{m}_i(H)$), $0 \leq i \leq n$, the cardinality of the smallest (respectively largest) monochromatic subset of vertices over all the $r_i(H)$ colorings of H with i colors.

Proposition 4.14. For any mixed hypergraph H the following inequalities hold:

$$m_i(H)i \leq n \leq \bar{m}_i(H)i, \; i = \chi, \chi+1, \ldots, \bar{\chi}.$$

Proof. This is evident.□

Consider a strict coloring of the mixed hypergraph H with t colors and let $Y \subseteq X$ be the largest set of vertices colored with the same color. Since any monochromatic subset is a stable set (i.e. it does not contain any edge as a subset), we have t stable sets which is a partition of X.

Take the first vertex from Y and choose one vertex of each color $2, 3, \ldots, t$. The set obtained is co-stable because it is polychromatic. Denote it by P_1. Choose the second vertex from Y and add one vertex of each of the remaining colors: we have a second co-stable set, P_2. We can repeat this procedure exactly $|Y|$ times. As a result we obtain a partition of X into $|Y|$ co-stable subsets $P_1, P_2, \ldots, P_{|Y|}$.

So, we can state the following

Proposition 4.15. If M_i, $1 \leq i \leq t$, $\max_i\{|M_i|\} = q$, are monochromatic subsets in any coloring of the mixed hypergraph H with t colors, then there exist q co-stable subsets P_j, $1 \leq j \leq q$ such that

$$\sum_{i=1}^{t} M_i = X = \sum_{j=1}^{q} P_j.$$

Proof. This is obvious.□

Proposition 4.16. For any mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$

$$\bar{\chi}(H) \leq \alpha_{\mathcal{A}}(H).$$

Proof. Take one vertex from each color set of X with $\bar{\chi}(H)$ colors to form a set A. Then A is co-stable. □

Mixed hypergraphs with $\bar{\chi}(H) = \alpha_{\mathcal{A}}(H)$ may be constructed easily, and it is seen now that $\alpha_{\mathcal{A}}(H)$ plays a role for $\bar{\chi}(H)$ analogous to that the maximal clique number plays for the chromatic number in a usual graph G [1].

Proposition 4.17. For any $k \geq 0$ and $\bar{\chi} \geq k+1$ one can construct a co-hypergraph H for which $\alpha_{\mathcal{A}}(H) - \bar{\chi}(H) > k$.

Proof. Let $H = (X, \mathcal{A})$ be the 3-uniform [1] co-hypergraph with $X = \{1, 2, \ldots, 2k+5\}$, and $A_1 = (1,2,3), A_2 = (1,4,5), \ldots, A_{k+2} = (1, 2k+4, 2k+5)$ such that

$$A_i \cap A_j = \{1\}, \quad i,j \in J = \{1, 2, \ldots k+2\}, \quad i \neq j.$$

It is easy to verify that $\alpha_{\mathcal{A}}(H) = 2k+4$ and $\bar{\chi}(H) = k+3$, hence $\alpha_{\mathcal{A}}(H) - \bar{\chi}(H) = k+1 > k$. □

Definition *4.18.[6] A mixed hypergraph H is called a co-perfect hypergraph if for all of its wholly-edge subhypergraphs H' the following equality holds:*

$$\bar{\chi}(H') = \alpha_{\mathcal{A}}(H').$$

Example 4.19. Call a vertex which is incident with at least one co-edge a partial vertex. Call a connected (by co-edges) maximal (with respect to inclusion) subgraph induced by partial vertices a co-component. Then any colorable mixed graph, in which every co-component induces a complete co-subgraph (each pair of vertices is a co-edge), is co-perfect.

Example 4.20. Any n-vertex r-uniform complete co-hypergraph K_n^r is co-perfect. Indeed, $\bar{\chi}(K_n^r) = \alpha_{\mathcal{A}}(K_n^r) = r - 1$.

Theorem *4.21. Any co-bistar $H = (X, \mathcal{A} \cup \mathcal{E})$ is a co-perfect mixed hypergraph.*

Proof. Obviously, $\bar{\chi}(H) = n - 1 = \alpha_{\mathcal{A}}(H)$. Now let the co-bitransversal be $\{x, y\}$. Consider any $Y \subseteq X$.

If H/Y contains at least one co-edge, then $\{x, y\} \subseteq Y$, hence H/Y is still a co-bistar and $\bar{\chi}(H/Y) = \alpha_{\mathcal{A}}(H/Y)$ by the above argument. Otherwise, we again have $\bar{\chi}(H/Y) = \alpha_{\mathcal{A}}(H/Y) = |Y|$. □

Denote by $\tau(H_{\mathcal{A}})$ the co-transversal number of a mixed hypergraph H [cf. 3, p.53], i.e. the cardinality of smallest subset of vertices that includes at least one vertex from every co-edge.

Definition *4.22. A mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$, $\mathcal{A} \neq \emptyset$, is called a co-monostar if the following conditions hold:*

$$1) \quad \tau(H_{\mathcal{A}}) = 1;$$

$$2) \quad \tau_2(H_{\mathcal{A}}) \neq 2.$$

In other words, a co-monostar is a mixed hypergraph which has exactly one vertex in common with each of its co-edges. A co-bistar has at least two such vertices. So, the classes of co-monostars and co-bistars have empty intersection.

It is evident that verifying conditions 1) and 2) may be implemented effectively in polynomial time. Note that the example in Proposition 4.17 is a co-monostar.

Theorem *4.23. Every co-monostar $H = (X, \mathcal{A} \cup \mathcal{E})$ is not a co-perfect mixed hypergraph.*

Proof. It follows from the definition of co-monostar that $\alpha_{\mathcal{A}}(H) = |X| - 1$. Since H is not a co-bistar, then by theorem 4.5 $\bar{\chi}(H) \neq |X| - 1$. Hence, $\bar{\chi}(H) \neq \alpha_{\mathcal{A}}(H)$.□

Example 4.24. A mixed hypergraph $\hat{H} = (X, \mathcal{A}^1 \cup \mathcal{E}^1)$ is called an *asymmetric complement* for a mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ if the following implications both hold:

$$1)\quad E_j \in \mathcal{E} \iff A_j^1 = X\backslash E_j \in \mathcal{A}^1,$$

$$2)\quad A_i \in \mathcal{A} \iff E_i^1 = X\backslash A_i \in \mathcal{E}^1.$$

Let C_k denote the usual simple cycle of length k, $k \geq 3$,. One can verify that the $\hat{C}_5$ is not co-perfect because $\bar{\chi}(\hat{C}_5) = 2$ and $\alpha_{_A}(\hat{C}_5) = 3$. However, the asymmetric complement of any cycle C_k, $k \geq 6$, is a co-perfect co-hypergraph. It may be represented as a union of two co-bistars opposite on C_k, and therefore $\bar{\chi}(\hat{C}_k) = n-2 = \alpha(\hat{C}_k)$.

Definition 4.25. *An r-uniform hypergraph $H = (X, \mathcal{E})$, $|X| = n \geq 3$, $r \geq 2$, is called a cycloid and denoted by C_n^r if $X = \{0, 1, \ldots, n-1\}$ and $\mathcal{E} = \{\{i, i + 1(mod\ n), \ldots, i + r - 1(mod\ n)\} : i = 0, 1, \ldots, n-1\}$.*

A co-cycloid can be similarly defined if $\mathcal{E}$ is replaced by $\mathcal{A}$ in the above definition.

In other words, one can say that for a cycloid there exists a graph $C_n = (X, V)$ representing a simple cycle without chords, such that $\mathcal{E}$ coincides with family of all paths of length $r-1$ on C_n.

Thus the usual cycle $C_n = C_n^2$ for any $n \geq 3$. Note that the example given by Algorithm 2 is the co-cycloid C_5^3.

Theorem 4.26. *If $C_n^r = (X, \mathcal{A})$ is a co-cycloid, $3 \leq r \leq n$, then the following implications hold:*

1) if $2r \leq n+1$, then C_n^r is not co-perfect;
2) if $2r \geq n+2$, then C_n^r is co-perfect.

Proof. 1) $2r \leq n+1$. If $2r \leq n$, then it is evident that all co-edges containing a fixed vertex generate a wholly-edge subhypergraph that is a co-monostar, and hence C_n^r is not co-perfect.

Suppose further that $2r = n+1$. Then C_n^r does not contain any co-monostar as a wholly-edge subhypergraph.

For $r = 3, 4$, it can be verified directly that C_n^r is not co-perfect. Hence let $r \geq 5$. Since $\tau(C_n^r) = 2$, it follows that $\alpha(C_n^r) = n-2$. We show that $\bar{\chi}(C_n^r) < n-2$. Assume, on the contrary, that $\bar{\chi}(C_n^r) = n-2$. Theorem 4.6 implies that C_n^r is either a union of two co-bistars with non intersecting co-bitransversals, or a hole. We consider these two cases:

a) C_n^r is a hole. Let x_1, x_2, x_3 be a bitransversal placed on C_n^r clockwise in this order, and let n_{ij} be the number of vertices between x_i and x_j, $i, j = 1, 2, 3$. We have $n_{12} + n_{23} + n_{31} + 3 = n = 2r - 1$. Since any co-edge must contain two vertices among x_1, x_2 and x_3, it follows that

$$n_{12} + 1 + n_{23} < r,$$

$$n_{23} + 1 + n_{31} < r,$$

$$n_{31} + 1 + n_{12} < r.$$

Summing these inequalities implies $r < 5$, a contradiction. Consequently, C^r_{2r-1} cannot be a hole for any $r \geq 3$.

b) C^r_n is a union of two co-bistars with non intersecting bitransversals. Assume that $x_1, x_2 \in X$ represent a bitransversal of the first co-bistar, and x_3, x_4 a bitransversal of the second co-bistar, and, moreover, x_1, x_2, x_3, x_4 are placed on C^r_n clockwise in this order. Let n_{ij} be the number of vertices between x_i and x_j, $i, j = 1, 2, 3, 4$. If $n_{12} = 0$ and $n_{34} = 0$, then a co-edge $A \in \mathcal{A}$ may be found easily such that $|A \cap \{x_1, x_2\}| \leq 1$ and $|A \cap \{x_3, x_4\}| \leq 1$; so assume that $n_{12} + n_{34} \geq 1$. We have $n_{12} + n_{23} + n_{34} + n_{41} + 4 = n = 2r - 1$. Since any co-edge must contain either $\{x_1, x_2\}$ or $\{x_3, x_4\}$, it follows that

$$n_{12} + n_{23} + n_{34} + 2 < r,$$

$$n_{12} + n_{14} + n_{34} + 2 < r.$$

By summing the above two inequalities, we have $2r - 1 + n_{12} + n_{34} < 2r$ which gives the contradiction $r < r$.

If the vertices x_1, x_2, x_3, x_4 are placed on C^r_n in any other order, then a co-edge $A \in \mathcal{A}$ may easily be found such that $|A \cap \{x_1, x_2\}| \leq 1$ and $|A \cap \{x_3, x_4\}| \leq 1$. Consequently, C^r_{2r-1} cannot be a union of two co-bistars for any $r \geq 3$. Thus the case 1) is proved.

2) $2r \geq n + 2$. Since for $r = n$ the theorem is evident, let $r \leq n - 1$. Hence, $\tau(C^r_n) = 2$, $\alpha(C^r_n) = n - 2$ and $|A_i \cap A_j| \geq 2$, $i, j \in I$. Let x_1 be a neighbor to x_2, x_3 be a neighbor to x_4, and the pair $\{x_1, x_2\}$ be opposite to the pair $\{x_3, x_4\}$ on C^r_n. $2r \geq n+2$ implies that for any $A_i \in \mathcal{A}$ either $|A_i \cap \{x_1, x_2\}| \geq 2$, or $|A_i \cap \{x_3, x_4\}| \geq 2$. Thus C^r_n is a union of two co-bistars with non intersecting co-bitransversals. It follows from Theorem 4.6 that $\bar{\chi}(C^r_n) = n - 2$. Since every wholly-edge subhypergraph of C^r_n in this case is co-perfect (because it represents a co-bistar), case 2) is proved.□

Definition 4.27. *A hypergraph $H = (X, \mathcal{E})$ is called (p, q, r)-Helly (cf. k-Helly, [3]) if for any subfamily of its edges the following implication holds:*

if every p elements of the subfamily have intersection of cardinality at least q, then the whole subfamily has intersection of cardinality at least r.

So, k-Helly hypergraphs from [3] are $(k, 1, 1)$-Helly. Call (2,2,2)-Helly hypergraphs **bi-Helly hypergraphs.**

Theorem 4.28. *Let $H = (X, \mathcal{A} \cup \mathcal{E})$, where $|X| = n \geq 6$ and $|E_i| \geq 3$ for $i \in J$, be a mixed hypergraph such that $H_{\mathcal{A}} = (X, \mathcal{A})$ is a bi-Helly co-hypergraph with $|A_i| \geq (n + 2)/2$, $i \in I$. Then H is co-perfect.*

Proof. From $n \geq 6$ and $|A_i| \geq (n + 2)/2$, $i \in I$, it follows that for any $i, j \in I$ $|A_i \cap A_j| \geq 2$. Since $H_{\mathcal{A}}$ is bi-Helly, $|\bigcap_{i \in I} A_i| \geq 2$. Hence H is a co-bistar and co-perfect by Theorem 4.21. □

Recall that a hypergraph $H = (X, \mathcal{E})$ is simple [3] if $E_i \subseteq E_j \Rightarrow i = j$.

Theorem 4.29. *A simple co-hypertree $H = (X, \mathcal{A})$ is co-perfect if and only if it does not contain any co-monostar as a wholly-edge subhypergraph.*

Proof. $\Rightarrow$ This is evident from Theorem 4.23.

$\Leftarrow$ Let $H = (X, \mathcal{A})$ be a simple co-hypertree without any co-monostar as a wholly-edge subhypergraph. Since every wholly-edge subhypergraph of a hypertree is also a hypertree, it is enough to show that we can color H with $\alpha(H)$ colors.

It is well known [1-3] that any hypertree satisfies the usual Helly property, i.e., for any $\mathcal{A}' \subseteq \mathcal{A}$ with $A_i \cap A_j \neq \emptyset$ for any pair $A_i, A_j \in \mathcal{A}'$ it follows that

$$\bigcap_{A_i \in \mathcal{A}'} A_i \neq \emptyset.$$

We show that $|\bigcap_{A_i \in \mathcal{A}'} A_i| \geq 2$ for any such $\mathcal{A}' \subseteq \mathcal{A}$. Assume that $\bigcap_{A_i \in \mathcal{A}'} A_i = \{x\}$. Consider the wholly-edge subhypergraph

$$H_1 = H / \bigcup_{A_i \in \mathcal{A}'} A_i = (\bigcup_{A_i \in \mathcal{A}'} A_i, \mathcal{A}_1),$$

where $\mathcal{A}' \subseteq \mathcal{A}_1$. Since H is a simple co-hypertree, any co-edge $A \in \mathcal{A}_1 \backslash \mathcal{A}'$ must contain the vertex x. Hence H_1 is a co-monostar, in contradiction to the theorem condition. Therefore $|\bigcap_{A_i \in \mathcal{A}'} A_i| \geq 2$ and H is thus a bi-Helly co-hypergraph, and moreover, $A_i \cap A_j \neq \emptyset$ implies $|A_i \cap A_j| \geq 2$ for any $A_i, A_j \in \mathcal{A}$.

Consider the problem of finding a minimal transversal [3] of H. For this, construct the intersecting (line) graph [3] of H, that is the graph $G = (\mathcal{A}, V)$, where $(A_i, A_j) \in V \Leftrightarrow A_i \cap A_j \neq \emptyset$. Since every clique of the graph G generates a co-bitransversal for the respective set of co-edges in H, a minimum covering of the graph G by cliques corresponds to a minimum covering of H by co-bitransversal of H. Let $C_1, C_2, \ldots, C_t$ be the cliques of such a minimal covering of G. Consequently, $\alpha_{\mathcal{A}}(H) = \alpha(H) = |X| - t$. Let the pair $\{x_i, y_i\}$ be the co-bitransversals corresponding to the cliques C_i, $i = 1, 2, \ldots t$. These all are different because $C_i \neq C_j$ implies that $\{x_i, y_i\} \cap \{x_j, y_j\} = \emptyset$. Color the vertices x_1, y_1 with the first color, x_2, y_2 with the second color,$\ldots, x_t, y_t$ with the t-th color, after that color all the remaining vertices each with a different color from $t+1, t+2, \ldots, |X| - t$. Thus we obtain a coloring of H with $|X| - t = \alpha(H)$ colors, and the theorem follows.□

5. Some problems and directions for future research.

This paper provides the beginning of Coloring Theory on mixed hypergraphs. We introduced the important notion of the upper chromatic number. Although any problem that was previously formulated for the lower chromatic number may be reformulated for the upper chromatic number, we believe that some new important and perspicacious problems and directions of research in this area could be the following ones [5,6,7]:

1. What is the upper chromatic number of co- and mixed hypergraphs without odd cycles, and of unimodular, balanced, arboreal, co-arboreal, normal, mengerian, paranormal [3, chapter 5] co- and mixed hypergraphs?

2. Investigate co-perfect co- and mixed hypergraphs and find all causes of non co-perfectness.

3. Is there any relationship between perfect graphs and uniform co-perfect co-hypergraphs?

4. Characterize those mixed hypergraphs with bounded breadth of their chromatic spectrum, i.e. with $b(H) \leq k$, in particular $b(H) = 1$ (mixed hypergraphs such that $\chi(H) = \bar{\chi}(H) > 0$).

5. Characterize uncolorable mixed hypergraphs, i.e. hypergraphs for which the splitting-contraction algorithm gives the empty list Z; it means that $\chi(H) = \bar{\chi}(H) = 0, R = (0, \ldots, 0)$.

Let $v(k), k \geq 0$, be the minimal number n such that there exists a minimal by inclusion uncolorable mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E}), \mid X \mid = n$, for which

$$\bar{\chi}(H_{\mathcal{A}}) - \chi(H_{\mathcal{E}}) = k.$$

What are $v(k), k = 0, 1, 2, \ldots$ equal to?

Consider an example for $k = 0$. Let $G = (X, V)$ be a graph with $X = \{1, 2, 3, 4, 5\}, V = \{(1,2), (1,3), (1,4), (1,5), (2,3), (3,4), (4,5)\}$. We have $\chi(G) = 3, r_3 = 1$. In the unique coloring of G with 3 colors the vertices 1,2,5 are all colored differently.

Let further $H_1 = (X, K_5^4 + A_0)$ be a co-hypergraph where $X = \{1, 2, 3, 4, 5\}, K_5^4$ is the 4-uniform complete hypergraph on 5 vertices and $A_0 = (1, 2, 5)$. We have $\bar{\chi}(H_1) = 3$, and in any coloring of H_1 with 3 colors at least two of vertices 1,2,5 are of the same color.

Construct the mixed hypergraph $H = (X, (K_5^4 + A_0) \cup V)$. It is uncolorable, and $\bar{\chi}(H_{\mathcal{A}}) - \chi(H_{\mathcal{E}}) = 0$. Consequently, $v(0) \leq 5$.

6. What are the bi-chromatic co-hypergraphs, i.e. co-hypergraphs with $\bar{\chi}(H) = 2$?

7. Characterize the hypergraphs with $O(H) \leq k,\ k \geq 0$, in particular, $O(H) = 0$.

8. Investigate the upper chromatic number of co- and mixed planar [2] hypergraphs, in particular, the opposite of the four coloring problem. Is it true that the upper chromatic number of any planar co-hypergraph without mono-stars is not less than $n - 4$?

9. Investigate the problems related to the upper chromatic number that are opposite to Hadwiger's problem [2].

10. Find the meaning of the chromatic polynomial's coefficients and roots for co- and mixed hypergraphs.

11. Investigate extremal problems [3, p.130] related to the upper chromatic number of co- and mixed hypergraphs.

12. What are the bi-Helly hypergraphs having a perfect (for example, chordal [1]) line graph?

13. Let H be a mixed hypergraph such that its dual hypergraph H^* [3, p.2] represents a (multi-) graph. In this case $\chi(H)$ and $\bar{\chi}(H)$ could be called the lower and upper chromatic indexes of a graph respectively. What are they equal to?

Vizing's theorem [1] resolves only the special case of this problem.

14. It is known in genetics [3, p.188] that hypertrees represent natural models for species of animals having common hereditary characteristics (populations). In

addition, hypertrees occupy a special place in co-hypergraphs colorings. Theorem 4.10 says that $O(H) = 0$ for any hypertree; this means that the greedy coloring algorithm may be applied without any re-coloring of vertices. Since in the case of time we cannot re-color the vertices (this would amount to "rescheduling the past"), this greedy strategy is the best for finding the longest life time.

Is it true that any population realizes the greedy co-hypergraph coloring algorithm in its life in order to achieve the longest life time and preserve hereditary characteristics?

Since hypertrees represent as much as the subclass of hypergraphs with $O(H) = 0$, may there exist other hereditary systems?

15. For a mixed hypergraph $H = (X, \mathcal{A} \cup \mathcal{E})$ call a hypergraph $\bar{H} = (X, \mathcal{A}_1 \cup \mathcal{E}_1)$ a chromatic inversion of H if $\mathcal{A}_1 = \mathcal{E}$, $\mathcal{E}_1 = \mathcal{A}$. What are the relations between chromatic numbers of H and $\bar{H}$, in particular, what are the coloring properties of mixed hypergraphs with $H = \bar{H}$?

CONJECTURE 1. The r-uniform co-hypergraph H is co-perfect if and only if it does not contain co-monostars and co-cycloids C^r_{2r-1}, $r \geq 3$, as wholly-edge subhypergraphs.

Theorem 4.29 shows that for co-hypertrees this conjecture is true.

CONJECTURE 2. For any sequence of positive numbers $N = (n_1, n_2, \ldots, n_t)$ such that $n_i \geq (n_{i-1} + n_{i+1})/2$, $i = 2, \ldots, n-1$, and $\max\{n_{[t/2]}, n_{[(t+2)/2]}\} = max_i\{n_i\}$ there exists such a mixed hypergraph H that $n_1 = r_{\chi}$, $n_2 = r_{\chi+1}$, $\ldots$, $n_t = r_{\bar{\chi}}$.

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