ON CONSTRUCTING GRAPHS WITH A PRESCRIBED ADJACENCY PROPERTY

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ABSTRACT:

A graph G is said to have property P(m,n,k) if for any disjoint sets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. We know that almost all graphs have property P(m,n,k). However, almost no graphs have been constructed. In this paper, we construct classes of graphs having property P(1,n,k). For the case m, $n \geq 2$, the problem of constructing graphs with the property P(m,n,k) seems difficult, with the only known examples being Paley graphs.

1. INTRODUCTION

For our purposes graphs are finite, loopless and have no multiple edges. For the most part our notation and terminology follows that of Bondy and Murty [8]. Thus G is a graph with vertex set V(G), edge set E(G), $\nu(G)$ vertices, $\varepsilon(G)$ edges, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. However, we denote the complement of G by $\bar{G}$.

A graph G is said to have property P(m,n,k) if for any disjoint sets A and B of vertices of G with $|A| = m$ and $|B| = n$ there exist at least k other vertices, each of which is adjacent to every vertex of A but not adjacent to any vertex of B. The class of graphs having property $P(m,n,k)$ is denoted by $\mathcal{G}(m,n,k)$. Observe that if $G \in \mathcal{G}(m,n,k)$, then $\bar{G} \in \mathcal{G}(n,m,k)$. The cycle C_ν of length ν is a member of $\mathcal{G}(1,1,1)$ for every $\nu \geq 5$. The well known Petersen graph is a member of $\mathcal{G}(1,2,1)$ and also of $\mathcal{G}(1,1,2)$. Despite these relatively simple examples few members of $\mathcal{G}(m,n,k)$ have been found. The class $\mathcal{G}(m,n,k)$ has been studied by: Ananchuen and Caccetta [2, 3]; Blass et. al. [5]; Blass and Harary [6]; Exoo [11]; Exoo and Harary [12, 13]. In addition, some variations of the above adjacency property have been studied by: Ananchuen and Caccetta [4]; Alspach et. al. [1]; Bollobás [7]; Caccetta et. al. [9] and Heinrich [14].

Blass and Harary [6] established, using probabilistic methods, that almost all graphs have property $P(n,n,1)$. From this it is not too difficult to show that almost all graphs have property $P(m,n,k)$. Despite this result, few graphs have been constructed which exhibit the property $P(m,n,k)$. An important graph in the study of the class $\mathcal{G}(m,n,k)$ is the so called **Paley graph** G_q defined as follows. Let $q \equiv 1 (\mathrm{mod}\ 4)$ be a prime power. The vertices of G_q are the elements of the finite field $\mathbb{F}_q$. Two vertices a and b are adjacent if and only if their difference is a quadratic residue, that is $a - b = y^2$ for some $y \in \mathbb{F}_q$.

For a prime $p \equiv 1 (\mathrm{mod}\ 4)$, Blass, Exoo and Harary [5] showed that $G_p \in \mathcal{G}(n,n,1)$ for $p > n^2 2^{4n}$. In [3] we improved this result by showing that for a prime power $q \equiv 1 (\mathrm{mod}\ 4)$, $G_q \in \mathcal{G}(n,n,k)$ for every $q > \{(2n - 3)2^{2n-1} + 2\} \sqrt{q} + (n + 2k - 1)2^{2n-1} - 2n^2 - 1$. Further, we proved that $G_q \in \mathcal{G}(m,n,k)$ for every

$$q > \{(t - 3)2^{t-1} + 2\} \sqrt{q} + (t + 2k - 1)2^{t-1} - 1,$$

where $t = m + n$; and $G_q \in \mathcal{G}(1,2,k)$ for every $q > (1 + 2\sqrt{2k})^2$. Computational results were also presented to determine the smallest Paley graphs in $\mathcal{G}(2,2,k)$ for small k.

In this paper, we will construct classes of graphs having property P(1,n,k). These classes include: the cubes; "generalized" Exoo-Harary graphs; "generalized" Petersen graphs; and "generalized" Hoffman - Singleton graphs. These graphs are described in Sections 2 and 3.

2. THE CLASS $\mathcal{G}(1,n,k)$, $n + k \le 4$

In this section we construct classes of graphs satisfying property P(1,n,k), $n + k \le 4$. We begin by stating the following two lemmas.

Lemma 2.1: (Ananchuen and Caccetta [2])

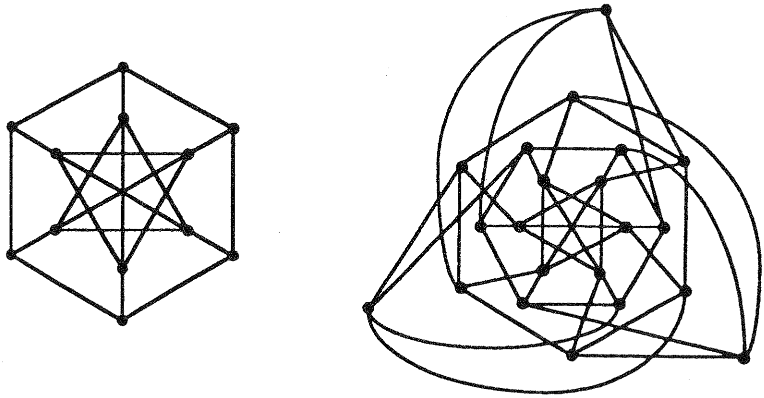
For $1 \le \ell \le n$, $\mathcal{G}(m,n,k) \subseteq \mathcal{G}(m,n - \ell,k + \ell)$. □

Lemma 2.2: (Exoo [11])

If G is a graph having girth at least 5 and $\delta(G) \ge n + k$, then $G \in \mathcal{G}(1,n,k)$. □

The work to follow makes use of the following notation. For a graph G we denote the neighbour set of a vertex u by N(u), and non-neighbour set of u by $\bar{N}(u)$. For vertices $u,v_1,v_2,\ldots,v_n$ of G we denote by $N(u/v_1,v_2,\ldots,v_n)$ the set of vertices of $V(G)\backslash\{u,v_1,v_2,\ldots,v_n\}$ which are adjacent to u but not adjacent to any v_i, $1 \le i \le n$.

Exoo and Harary [13] constructed small graphs in the class $\mathcal{G}(1,n,1)$. In particular, they proved that the graphs G_1 and G_2 displayed in Figures 2.1 are smallest order graphs of girth 3 in the classes $\mathcal{G}(1,2,1)$ and $\mathcal{G}(1,3,1)$, respectively.



G_1 $\qquad$ G_2

Figure 2.1

We now generalize these graphs. Let $C = u_0u_1 \ldots u_{m-1}u_0$ be a cycle of length m, $m \geq 5$. For s a positive integer, define the graph $E(m,s)$ as follows. Take $\bar{C}$ and s copies $C_0, C_1, \ldots, C_{s-1}$ of C. Join vertex u_i, of C_j, $0 \leq i \leq m-1$, $0 \leq j \leq s-1$, to vertex u_i of $\bar{C}$. $E(m,s)$ is the resulting graph. Observe that $E(5,1)$ is the Petersen graph and the graph G_1 of Figure 2.1 is $E(6,1)$. It is easy to see that $E(m,s)$ is a graph on $(s+1)m$ vertices having minimum degree 3. The adjacency properties of $E(m,s)$ are given in the following result.

Theorem 2.1: For any positive integers $m \geq 5$ and s, $E(m,s) \in \mathcal{G}(1,n,k)$, for $n + k \leq 3$.

Proof: In view of Lemma 2.1 it is sufficient to show that $E(m,s) \in \mathcal{G}(1,2,1)$. Let u be any vertex of $E(m,s)$. If $u \in C_j$ for some j, then u does not belong to any cycle of length less than 5. Hence for any other vertices v and w, $N(u/v,w) \neq \phi$. If, on the other hand, $u \in \bar{C}$ at least $s + 2 \geq 3$ vertices $a_1, a_2, \ldots, a_{s+2}$ are needed so that $N(u) \subseteq \bigcup_{i=1}^{s+2} N(a_i)$. Hence for any distinct pair of vertices v and w of $E(m,s)-u$, $N(u/v,w) \neq \phi$. Therefore $E(m,s) \in \mathcal{G}(1,2,1)$, as required. □

We next generalize the graph G_2 displayed in Figure 2.1. Let $m = 2r \geq 6$ and $2 \leq s \leq r$. Define the graph $E^*(2r,s)$ by adding r new vertices labelled $0,1,2,\ldots,r-1$ to $E(2r,s)$ and the 2rs edges:

$$\{(i,u_{i+j}),\ (i,u_{i+j+r}) : u_{i+j},\ u_{i+j+r} \in C_j,\ 0 \leq i \leq r-1,\ 0 \leq j \leq s-1\};$$

all subscripts are read modulo 2r. Note that the G_2 graph in Figure 2.1 is just $E^*(6,2)$. It is easy to see that $E^*(2r,s)$ is a graph on $(2s+3)r$ vertices having minimum degree 4. The adjacency properties of $E^*(2r,s)$ are given in the following result.

Theorem 2.2: For $r \geq 3$ and $2 \leq s \leq r$, $E^*(2r,s) \in \mathcal{G}(1,n,k)$, for $n + k \leq 4$.

Proof: In view of Lemma 2.1 it is sufficient to show that $E^*(2r,s) \in \mathcal{G}(1,3,1)$. Let u be any vertex of $E^*(2r,s)$. If u is a new vertex added to $E(2r,s)$ or $u \in C_j$ for some j, then every edge incident to u belongs to a cycle of length 5 and to no smaller cycle. Hence for any other distinct vertices x, v and w, $N(u/x,v,w) \neq \phi$. If, on the other hand, $u \in \bar{C}$, then at least $s + 2 \geq 4$ vertices $a_1, a_2, \ldots, a_{s+2}$ are needed so that

$N(u) \subseteq \bigcup_{i=1}^{s+2} N(a_i)$. Hence for any distinct vertices x, v and w of $E^*(2r,s)-u$, $N(u/x,v,w) \neq \phi$. Therefore $E^*(2r,s) \in \mathcal{G}(1,3,1)$, as required. □

We noted in the introduction that the Petersen graph is in the class $\mathcal{G}(1,2,1) \cap \mathcal{G}(1,1,2)$. We now generalize this graph to give another class of graphs in $\mathcal{G}(1,n,k)$, $n + k \leq 4$.

Let m and t be integers, $m \geq 5$ and $m \neq 6$, satisfying

$$1 < t < m - 1, \qquad m \neq 2t. \tag{2.1}$$

Define the graph $I(m,t)$ as follows. The vertices of $I(m,t)$ are $\{v_0, v_1, \ldots, v_{m-1}\}$ and the edges are $\{(v_i, v_{i+t}) : 0 \leq i \leq m - 1\}$; all subscripts are read modulo m. Noting that $I(m,t) \cong I(m,m - t)$, we can replace (2.1) with

$$1 < t < \tfrac{1}{2}m . \tag{2.2}$$

We use the normal convention of denoting the greatest common divisor of integers a and b by (a,b). Observe that $I(m,t)$ consists of exactly (m,t) disjoint cycles each of length $m/_{(m,t)}$. Furthermore, for each m we can choose at least one value t_0 of t such that $m/_{(m,t_0)} \geq 5$. Note that $t_0 \neq 1$.

Define the graph $G(m,t_0,s)$ as follows. Start with the graph $I(m,t_0)$ and s copies $C_0, C_1, \ldots, C_{s-1}$ of the m-cycle $C = u_0u_1\ldots u_{m-1}u_0$. Add the ms edges:

$$\{(v_i, u_i) : v_i \in I(m,t_0),\ u_i \in C_j,\ 0 \leq i \leq m - 1,\ 0 \leq j \leq s - 1\}.$$

Note that $G(5,2,1)$ is the Petersen graph. Observe that the graph $G(m,t_0,s)$ has $(s + 1)m$ vertices, girth at least 5, minimum degree 3. Therefore, by lemmas 2.1 and 2.2 we have:

Theorem 2.3: Let m and t_0 be integers, $m \geq 5$ and $m \neq 6$ and $m/_{(m,t_0)} \geq 5$. Then $G(m,t_0,s) \in \mathcal{G}(1,n,k)$, for $n + k \leq 3$. □

Remark 1: When $I(m,t_0) \cong I(m,t_0')$, $t_0 \neq t_0'$ it is still possible for $G(m,t_0,s) \not\cong G(m,t_0',s)$. For example, $I(11,2) \cong I(11,3)$ but $G(11,2,1) \not\cong G(11,3,1)$.

For $r \geq 4$, $2 \leq s \leq r$, we construct the graph $G^*(2r,t_0,s)$, from $G(2r,t_0,s)$ as follows. Add r new vertices labelled $0,1,2,\ldots,r - 1$ to $G(2r,t_0,s)$ and the 2rs edges:

$$\{(i,u_{i+j}),\ (i,u_{i+j+r}) : u_{i+j},\ u_{i+j+r} \in C_j,\ 0 \leq i \leq r-1,\ 0 \leq j \leq s-1\};$$

all subscripts are read modulo 2r. It is easy to see that $G^*(2r,t_0,s)$ is a graph on $(2s + 3)r$ vertices having minimum degree 4. The adjacency properties of $G^*(2r,t_0,s)$ are given in the following result.

Theorem 2.4: For $r \geq 4$, $2 \leq s \leq r$ and $2r/_{(2r,t_0)} \geq 5$, $G^*(2r,t_0,s) \in \mathcal{G}(1,n,k)$, for $n + k \leq 4$.

Proof: In view of lemmas 2.1 and 2.2, we need only to show that $G^*(2r,t_0,s)$ has girth at least 5. From the construction of the graph $G^*(2r,t_0,s)$ it is sufficient to show that this graph contains no 4-cycle. Since $G(2r,t_0,s)$ has girth at least 5 and $t_0 \neq 1$, without any loss of generality we can assume that $G^*(2r,t_0,s)$ contains a 4-cycle consisting of one new vertex and one vertex from each of $C_j, C_h (h \neq j)$ and $I(2r,t_0)$. This is clearly impossible. □

Remark 2: For $m \geq 9$, the graph $\hat{G}(m,t_0,1)$ obtained from $G(m,t_0,1)$ by adding a vertex x and the edges xu_0, xu_3 and xu_6 is in the class $\mathcal{G}(1,2,1)$. In fact, for any $G \in \mathcal{G}(1,2,1)$, we can add a vertex x of degree at least 3 provided the distance between the vertex of N(x) is at least 3 and the result is a graph in $\mathcal{G}(1,2,1)$.

3. $\mathcal{G}(1,n,k)$

In this section we will show that the t-cube is in the class $\mathcal{G}(1,n,k)$, for $2n + k \leq t$. Further, for any n and k, we construct a class of graphs with property P(1,n,k). We begin with the cube.

The **t-cube**, Q_t, is defined as follows: the vertices of Q_t are the 2^t vectors $(e_1,e_2,\ldots,e_t)$ where $e_i = 0$ or 1, $i = 1,2,\ldots,t$ and two vertices are adjacent if and only if their symbols differ in exactly one coordinate. Thus Q_t is a t-regular graph on 2^t vertices.

Exoo and Harary [12] proved that $Q_{2n+1} \in \mathcal{G}(1,n,1)$. A more general result is:

Theorem 3.1: $Q_t \in \mathcal{G}(1,n,k)$, for any $2n + k \leq t$.

Proof: Let x be a vertex of Q_t and $u_1,u_2,\ldots,u_t$ be the neighbours of x. From the definition of a cube we know that Q_t contains no triangle and no vertex of Q_t except x is adjacent to more than two vertices of the u_i, $i = 1,2,\ldots,t$. Since $2n + k \leq t$, for any n-set A of vertices of Q_t-x there are at least k other vertices, each of which is adjacent to x but not adjacent to any vertex of A, as required. □

Our next class of graphs comes from generalizing the Hoffman-Singleton graphs.

Let $p \geq 5$ be a prime and t an integer satisfying $1 < t < \frac{1}{2}p$. For $m = p$ the graph $I(m,t)$ defined in Section 2 is a cycle of length p. For $1 \leq r, s \leq p$ we define the graph $H_s^r(p,t)$ as follows. Take r copies $I_0, I_1, \ldots, I_{r-1}$ of $I(p,t)$ and s copies $C_0, C_1, \ldots, C_{s-1}$ of the p-cycle $C = u_0u_1\ldots u_{p-1}u_0$. Add the prs edges:

$$\{(v_i, u_{i+jk}) : v_i \in I_j,\ u_{i+jk} \in C_k,\ 0 \leq i \leq p-1,$$
$$0 \leq j \leq r-1,\ 0 \leq k \leq s-1\}.$$

Note that for the particular case $p = 5$, $t = 2$ we have (see [15]): the Petersen graph $H_1^1(5,2)$; the Wegner graph $H_3^3(5,2)$; the O'Keefe-Wong graph $H_4^4(5,2)$ and the Hoffman-Singleton graph $H_5^5(5,2)$. Observe that the graph $H_s^r(p,t)$ has $(r + s)p$ vertices, minimum degree $\delta = 2 + \min\{r,s\}$. Of course, $H_r^r(p,t)$ is $(r + 2)$-regular. The adjacency properties of $H_s^r(p,t)$ are given in the following theorem.

Theorem 3.2: Let $p \geq 5$ be a prime, $1 < t < \frac{1}{2}p$ and $1 \leq r, s \leq p$. Then $H_s^r(p,t) \in \mathcal{G}(1,n,k)$, for $n + k \leq 2 + \min\{r,s\}$.

Proof: In view of lemmas 2.1 and 2.2 we need only show that $H_s^r(p,t)$ has girth at least 5. From the construction of $H_s^r(p,t)$ it is sufficient to show that the graph contains no 4-cycle. So suppose $v_iu_kv_ju_\ell v_i$ is a 4-cycle in $H_s^r(p,t)$. Without any loss of generality assume that $v_i \in I_a$, $v_j \in I_b$, $u_\ell \in C_c$ and $u_k \in C_d$, where $a \neq b$ and $c \neq d$. Then

$$\ell \equiv i + ac \equiv j + bc \pmod{p}, \text{ and}$$
$$k \equiv i + ad \equiv j + bd \pmod{p}.$$

Therefore $ac - ad \equiv bc - bd \pmod p$ and thus $(a - b)(c - d) \equiv 0 \pmod p$. Hence $a \equiv b \pmod p$ or $c \equiv d \pmod p$. Now, since $0 < a, b, c, d < p$ and p is a prime, we have equality, a contradiction. This completes the proof of the theorem. □

We conclude this paper by constructing a graph $G \in \mathcal{G}(1,2,1)$ on n vertices for every integer $n \geq 10$ and $n \neq 11$. As noted in remark 2 the graph $\hat{G}(m,t_0,1)$, $m \geq 9$ together with Theorem 2.3 establishes the existence of G for n = 14, 15, 16 and for $n \geq 18$. For n = 10 we have the Petersen graph whilst for n = 12 and 13 we can take the graphs displayed in Figure 3.1. For n = 17 we can take the Paley graph G_{17}. We remark that it is not to difficult to show that $\mathcal{G}(1,2,1) = \phi$ for every $n \leq 9$ and $n = 11$.

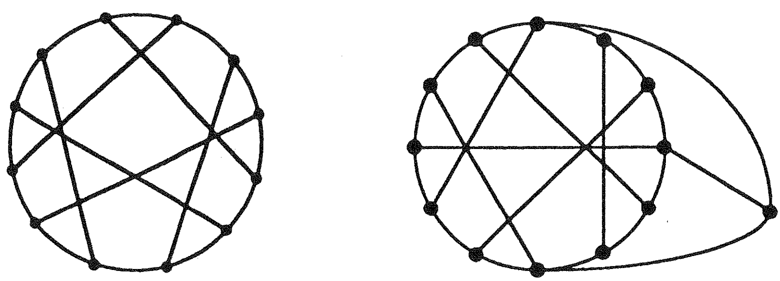


Figure 3.1

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