Multiplication of sequences with zero autocorrelation

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Abstract

Near normal sequences of new lengths $n = 4m + 1 = 49, 53, 57$ are constructed. The relation between a special set of near normal sequences and Golay sequences is discussed. A reformulation of Yang's powerful theorems on T-sequences is also given.

We give base sequences for lengths $m + p$, $m + p$, m, m for $p = 1$ and $m \in \{19, ..., 30\}$. Some of these are new lengths, or new decompositions into four squares for n and constructed here for the first time.

1 Introduction

We use the notations and definitions of Koukouvinos, Kounias, Seberry, C.H. Yang and J. Yang [9] except for those few noted below.

If $A = \{a_1, ..., a_n\}$, $B = \{b_1, ..., b_n\}$ are sequences of length n, we use the notation $A^* = \{a_n, ..., a_1\}$ for the reversed sequence, A/B for the sequence $\{a_1, b_1, ..., a_n, b_n\}$.

If $A = \{a_1, ..., a_{n+1}\}$, $B = \{b_1, ..., b_n\}$ we use the notation A/B for the sequence $\{a_1, b_1, ..., a_n, b_n, a_{n+1}\}$ and

$$N_{AB}(s) = \sum_{i=1}^{n-s} a_i b_{i+s}, \quad s = 0, 1, ..., n-1$$

for the cross-correlations.

Definition 1 A quadruple $(E, F; G, H)$ of $(0, \pm 1)$ sequences is said to be a set of *near normal sequences* for length $n = 4m + 1$ (abbreviated as NN(n)) if the following conditions are satisfied.

(i) $E = (X/O_{m-1}, 1)$, $F = (Y/O_{m-1}, 0)$ where X and Y are ± 1 sequences of length m, i.e. E and F are of length $2m$. G and H are $(0, \pm 1)$ sequences of length $2m$, such that $G + H$ is a ± 1 sequence of length $2m$.

(ii) $N_E(s) + N_F(s) + N_G(s) + N_H(s) = 0, \; s = 1, ..., 2m - 1$.

Condition (ii) is also equivalent to

$$E(z)E(z^{-1}) + F(z)F(z^{-1}) + G(z)G(z^{-1}) + H(z)H(z^{-1}) = 4m + 1, \; z \neq 0. \quad (1)$$

Remark 1 It is easy to see that the sequences G and H of Definition 1 are quasi-symmetric, i.e. if $g_k = 0$, then $g_{2m+1-k} = 0$ and also if $h_k = 0$, then $h_{2m+1-k} = 0$ (see [9]).

Let A, B, C, D be base sequences of lengths $n+1, n+1, n, n$ with $a_0 = b_0 = 1$, and let $X = (A+B)/2 = (1, U, 0)$, $Y = (A-B)/2 = (0, V, 1)$, $Z = (C+D)/2$ and $W = (C-D)/2$, then pairs U, V and Z, W are both quasi-symmetric supplementary sequences ie. $U \pm V$ and $Z \pm W$ are both sequences of $1, -1$. If $V = 0_{n-1}$, the zero sequence of length $n-1$, then $F = (1, U)$, $G = Z$ and $H = W$ are NS(n). Also if $n = 2m$ and $U = O_m/u$, $V = v/O_{m-1}$, then $E = (V, 1)$, $F = (1, U')$, $G = Z$ and $H = W$, where $U' = O_{m-1}/u$, are NN($4m + 1$).

It is easy to see that NS(n), $F; G, H$ are equivalent to base sequences $F, F, G + H, G - H$ of lengths n, n, n, n. Consequently quasi-symmetry follows from zero autocorrelation.

Therefore both NS and NN are obtainable from the special sets of base sequences stated above. Thus the quasi-symmetry of NN is a consequence of zero autocorrelation. These special base sequences are essential for the composition of four complementary sequences.

In this paper we construct near normal sequences NN(n), of new lengths $n = 4m+1 = 49, 53, 57$ and base sequences of lengths $n+1, n+1, n, n$ for $n = 19, 20, ..., 30$. We discuss the relation between a special set of near normal sequences and Golay sequences. Finally we give a reformulation of Yang's powerful theorems on T-sequences and summarize the known results on Yang numbers, base sequences and T-sequences.

2 On Golay sequences and near normal sequences

One of us (C.Yang [15]) has noted the following result. However Eliahou, Kervaire and Saffari [3] have shown that this construction cannot produce previously unknown Golay sequences.

Theorem 1 *A special set of near normal sequences $NN(n)$, $n = 4m+1$, $(E, F; G, O_{2m})$ with "symmetric" F and skew G are basic sequences from which Golay sequences $GS(2n)$ can be built.*

Proof. By letting $C(z) = G(z^2) + zF(z^2)$ and $D(z) = E(z^2)$, we have

$$\begin{aligned} C(z)C(z^{-1}) + D(z)D(z^{-1}) &= G(z^2)G(z^{-2}) + F(z^2)F(z^{-2}) + E(z^2)E(z^{-2}) \\ &= n, \quad z \neq 0 \end{aligned}$$

and

$$\begin{aligned} A(z) &= C(z) + z^3 D(z) \\ B(z) &= C(z) - z^3 D(z) \end{aligned}$$

satisfy

$$A(z)A(z^{-1}) + B(z)B(z^{-1}) = 2n, \quad z \neq 0.$$

Consequently

$$\begin{aligned} L(z) &= A(z) + z^{n-1}B^*(z) \quad \text{and} \\ M(z) &= A(z) - z^{n-1}B^*(z) \end{aligned}$$

are GS($2n$), i.e.

$$L(z)L(z^{-1}) + M(z)M(z^{-1}) = 2(2n), \quad z \neq 0.$$

□

Example 1 (i) $n = 5$: $E = (++)$, $F = (+0)$; $G = (+-)$, $H = (00)$. From Theorem 1 we obtain: $C = (++-)$, $D = (+0+)$, $A = (++-+0+)$, $B = (++--0-)$ and so the sequences $L = (++-+-+--++)$, $M = (++-+++++--)$ are GS(10).

(ii) $n = 13$:

$$\begin{aligned} &E = (+0-0++), \quad F = (+0+0+0); \\ &G = (++-+--), \quad H = (000000). \end{aligned}$$

From Theorem 1 we obtain:

$$\begin{aligned} C &= (+++0-++0-+-) \\ D &= (+000-000+0+) \end{aligned}$$

and

$$\begin{aligned} A &= (++++-++--+-+0+) \\ B &= (+++--+++-+--0-) \end{aligned}$$

and so the sequences

$$\begin{aligned} L &= (++++-++--+-+-+--+-+++--+++) \\ M &= (++++-++--+-+++++-+---++---) \end{aligned}$$

are GS(26).

3 On base sequences and near normal sequences

We formally prove the result of Yang [14]:

Theorem 2 *$(E,F;G,H)$, $n=4m+1$ where $E=(X/0_{m-1},1)$ and $F=(Y/O_{m-1},0)$, being near normal sequences NN(n), is equivalent to the ±1 sequences $A=(Y/X,1)$, $B=(Y/(-X),-1)$, $C=G+H$, $D=G-H$ of lengths $2m+1, 2m+1, 2m, 2m$ respectively being base sequences BS($4m+1$).*

Proof. Write

$$\begin{aligned} A(z) &= F(z^2)+zE(z^2) \\ B(z) &= F(z^2)-zE(z^2) \\ C(z) &= G(z)+H(z) \\ D(z) &= G(z)-H(z). \end{aligned} \tag{2}$$

From (2) we obtain

$$\begin{aligned} & A(z)A(z^{-1})+B(z)B(z^{-1})+C(z)C(z^{-1})+D(z)D(z^{-1}) \\ & \quad = (F(z^2)+zE(z^2))(F(z^{-2})+z^{-1}E(z^{-2}))+(F(z^2)-zE(z^2))(F(z^{-2})-z^{-1}E(z^{-2})) \\ & \quad +(G(z)+H(z))(G(z^{-1})+H(z^{-1}))+(G(z)-H(z))(G(z^{-1})-H(z^{-1})) \\ & \quad = 2(E(z^2)E(z^{-2})+F(z^2)F(z^{-2})+G(z)G(z^{-1})+H(z)H(z^{-1})) \\ & \quad = 8m+2 \end{aligned}$$

i.e. A,B,C,D are BS($4m+1$).

The proof of the converse is straightforward. □

The algorithm described in [8] can now be modified to give a special set of base sequences A,B,C,D of lengths $n+1, n+1, n, n$ as indicated in Theorem 2 to find near normal sequences NN($2n+1$). This modified algorithm was used to find the results in Tables 1 and 2. One of us (Joel Yang) has found a different algorithm (unpublished) for computing base sequences which gives us a method of independently verifying our results.

The near normal sequences given in Table 1 for $n=61$ are due to C. Yang [15]. The base sequences of length 61 constructed from these are given in Table 2.

Length n	Sums of squares $a^2+b^2+c^2+d^2$	Sequences
49	$7^2+0^2+0^2+0^2$	There are no near normal sequences.
49	$3^2+2^2+6^2+0^2$	$E = (-0-0-0-0-0-0+0+0-0+0+0-+)$ $F = (+0+0+0-0-0-0+0+0-0+0-0+0)$ $G = (00+0++0+0--00++0-0++0+00)$ $H = (+-0+00+0-00+-00+0-00-0-+)$
49	$5^2+2^2+4^2+2^2$	$E = (+0+0+0+0+0+0-0-0+0-0-0++)$ $F = (+0+0+0-0-0-0+0+0-0+0-0+0)$ $G = (++0-00+0-00--00-0+00-0--)$ $H = (00+0+-0-0+-00-+0-0+-0-00)$
49	$1^2+4^2+4^2+4^2$	$E = (-0-0+0-0+0+0+0+0-0-0-0++)$ $F = (+0+0+0+0-0+0-0+0+0-0-0+0)$ $G = (+0++00++-++00--+-+00--0+)$ $H = (0+00-+00000++00000-+00+0)$
53	$4^2+1^2+6^2+0^2$	$E = (+0-0-0+0-0-0-0+0+0-0-0-0-+)$ $F = (+0+0+0+0-0+0-0-0+0-0-0-0+0)$ $G = (-0+00+0+++00-+00-++0+00-0+)$ $H = (0-0+-0+000++00-+000-0--0+0)$
57	$1^2+2^2+6^2+4^2$	$E = (-0+0+0+0+0+0-0-0-0+0-0-0+0-+)$ $F = (+0-0+0+0-0+0-0+0+0-0-0+0+0-0)$ $G = (++0+0+0+++--0+-0+++-+0-0-0-+)$ $H = (00-0-0+00000+00-00000-0-0-00)$
61	$2^2+5^2+4^2+4^2$	$E = (-0-0-0-0-0+0-0-0+0+0+0-0+0-0++)$ $F = (+0+0+0+0-0-0+0-0+0+0+0+0-0-0+0)$ $G = (--00-++0+0-++0+-0+++0+0--+00-+)$ $H = (00+-000+0+000-00+000-0+000++00)$

Table 1: Near normal sequences $NN(n)$: $(E, F; G, H)$, $n = 4m+1$

Length n	Sums of squares $a^2+b^2+c^2+d^2$	Sequences
39	$0^2+2^2+5^2+7^2$	$A=(--++++++--++-+--+-+-)$ $B=(--+---+--+-+++++--++)$ $C=(+++++---+++-+-++--+)$ $D=(++-+++--+-+-+++++-+)$
39	$8^2+2^2+3^2+1^2$	$A=(+-++++++-++-+-+-+++-)$ $B=(+++++---+--+-+--++-+)$ $C=(-++++--+++++--+---+)$ $D=(---+++--+-+--++-+++)$
41	$7^2+5^2+2^2+2^2$	$A=(+++++++-+-+--+--+++-+)$ $B=(-+-+-+------+++--+--+)$ $C=(++++-+--++++--+-+---)$ $D=(+---++--++++--++-++-)$
41	$9^2+1^2+0^2+0^2$	$A=(+-+++--+-++-++++-++++)$ $B=(---+--++++---+-+++-++)$ $C=(-++++++---+-+----++-)$ $D=(--++--+-+-+++-++-+--)$
41	$5^2+5^2+4^2+4^2$	$A=(+---++-+-+++++++---++)$ $B=(--+--+++++-+-+-++-+++)$ $C=(++--++--+-++-+--++++)$ $D=(++---+-+-+--++-+++++)$
41	$1^2+7^2+4^2+4^2$	$A=(+--++++-+-+---+-++--+)$ $B=(--++-+------+----++-+)$ $C=(++++--+-+-++-+++---+)$ $D=(++-----++-+++-++++-+)$
43	$2^2+0^2+9^2+1^2$	$A=(-++--+-+--+++---+++++-)$ $B=(-+-+----++++-++--+--++)$ $C=(-+++-+-+-++++++++-++-)$ $D=(--+-+--+-++++++--++--)$
43	$8^2+2^2+3^2+3^2$	$A=(+------+-+-+---++----+)$ $B=(+--+-++++++--+----+---)$ $C=(+-++--++-++-+-+---+++)$ $D=(--+-++--+++--+-+++++-)$
43	$6^2+0^2+7^2+1^2$	$A=(++--++--+--++++++-+-++)$ $B=(++--++++-++-----+-+-+-)$ $C=(-+-+-++-++-++++--++++)$ $D=(--++----++++++-+-+--+)$
45	$9^2+3^2+0^2+0^2$	$A=(-++++-++-+-+++++-+++-+-)$ $B=(+++++-+---++--+--+++-+-)$ $C=(++++-++-----+-++---+-+)$ $D=(++++---++---++--+--+-+)$
45	$3^2+1^2+8^2+4^2$	$A=(++++-+---++-+---++--+++)$ $B=(--+++--+--+-++-++-+-+-+)$ $C=(+++++++++-+-+-+--+++--)$ $D=(---++++++--++-+--++-++)$
45	$9^2+1^2+2^2+2^2$	$A=(++++-+++-+-+++++---+-++)$ $B=(+-+-++++-++--+++--+----)$ $C=(+-+---++-+-++++--+----)$ $D=(+-+--+--++-++--+++----)$

Table 2: Base sequences BS(n): $(A, B; C, D)$ with lengths $m+1, m+1, m, m$, where $n = 2m+1$ and $2n = a^2+b^2+c^2+d^2$.

Length $n = 2m + 1$	Sums of squares $a^2 + b^2 + c^2 + d^2$	Sequences
45	$1^2 + 7^2 + 6^2 + 2^2$	$A = (+ - + - - - - + - - + + + + - + + - - + + - +)$ $B = (- + + - + + - - - + + + + - - - + + + + + + +)$ $C = (+ - + + - - + + + + + + + + - + - + - - - +)$ $D = (- + + - + - + - + + - - + + + + - - + - + -)$
47	$0^2 + 2^2 + 9^2 + 3^2$	$A = (+ + + - - - - - + - - + + - - + + + - - + + - +)$ $B = (+ + + + - - - + + - + - - + - + - + - - - + - -)$ $C = (+ + + + - - + - - - + + + + + - + + + + + - +)$ $D = (- - - + + - + + - + + - + - + + + + - + - + -)$
47	$0^2 + 6^2 + 7^2 + 3^2$	$A = (+ - + + - + - + + + - - + + - - + - - + + - - -)$ $B = (+ - + - - + - - + + + + - - - - - - - + - - - +)$ $C = (+ + + - + + + + - - + - + + + + + - - - + - +)$ $D = (+ - + + - + - + - + - - - - + + - - + + + + +)$
49	$7^2 + 7^2 + 0^2 + 0^2$	$A = (+ + - - - + - + - + - + + + - + + - - + + + + + +)$ $B = (+ + + + - + + - - - + + - + + - + + + + + - - + -)$ $C = (+ + + + + + - - - + - + - - + - - - - + - + + -)$ $D = (+ - + - - + - + + + - + - - + + + - - - + + - -)$
49	$1^2 + 5^2 + 6^2 + 6^2$	$A = (+ - + - + - - - - - - - + + + + - - + + - + + - +)$ $B = (- - - - - - + - + - + - - + - + + - - + + + - - +)$ $C = (+ - + + + + + + - - - + - + + + - - + + - + - +)$ $D = (- + + - + + - + + - - - + + + - - + + + + + + -)$
49	$7^2 + 3^2 + 6^2 + 2^2$	$A = (+ + + + + + - + - + - + + - + - - + + - - - + + +)$ $B = (- + - + - + + + + + + + - - - - + + - - + - - + +)$ $C = (+ + + - + - + - - + - - - - + - - + + - - - - -)$ $D = (+ + - - - + + + - - + - - + - - + + - + - + - -)$
49	$5^2 + 3^2 + 8^2 + 0^2$	$A = (+ - + - + + + - - + + + - + + + + - - - - - + + +)$ $B = (- - - - - + - - + + - + + + - + - - + - + - - + +)$ $C = (+ + + + - + + + - + + + + - - + - + - + - - + +)$ $D = (+ - + + + - + + - + + - - - - + - + + - - - - +)$
51	$8^2 + 6^2 + 1^2 + 1^2$	$A = (+ + + + + + + + - - + + + - - + - - + + - + - + - +)$ $B = (- + + + + + + + - - + + + - - + - - + + - + - + - +)$ $C = (+ + + + + + - - - - + + - - + + - + - - + - + - -)$ $D = (+ - + - + - - + - + + - - + + - - - - + + + + + -)$
53	$5^2 + 3^2 + 6^2 + 6^2$	$A = (+ - - - + + - - - + - + + + + + - + - - + + - + + + +)$ $B = (- - - - + + - - - + - + + + + + - + - - + + - + + + +)$ $C = (+ + + - - + + + - + - - + - + - + - - + + - + + + +)$ $D = (+ + + - - + + + - + - - + - + - + - - + + - + + + +)$
53	$3^2 + 5^2 + 6^2 + 6^2$	$A = (- - - + - + - - + + - + + + + - - - + + + + + + - + -)$ $B = (+ - + + + + + - - + + + - + - - + - - + - + - + + + -)$ $C = (- - + + - + + + + + + + - + - + - + + - + - - - + +)$ $D = (- + + - + + - + + + - - - + + - - + + + + + + - - +)$
55	$0^2 + 10^2 + 1^2 + 3^2$	$A = (- + - + - - - - + + + + + + - + + - - + - + - - + + - -)$ $B = (- - + + + + - + + + - - + + - + - + - + + + + + + - + +)$ $C = (+ + + - + + - - - - + + + + - - + - - + - + - + - - +)$ $D = (- + - + + + - + + + + + + - - - + + + - - + - - + - -)$
55	$8^2 + 6^2 + 3^2 + 1^2$	can be obtained by changing the signs of even elements of each sequence

Table 2 (cont.): Base sequences BS(n): $(A, B; C, D)$ with lengths
$m + 1, m + 1, m, m,$
where $n = 2m + 1$ and $2n = a^2 + b^2 + c^2 + d^2$.

Length n	Sums of squares $a^2+b^2+c^2+d^2$	Sequences
55	$4^2+2^2+3^2+9^2$	$A=(-++-++---++++----+-+-+++-+++)$ $B=(+++-+-++-+-+---+---++--+-+++)$ $C=(++-+-+++++--+-++--+++--+---)$ $D=(++-++-+++-++-+--++++++++---)$
57	$3^2+1^2+2^2+10^2$	$A=(+--+++++-+++--+-+--+--+-++--+)$ $B=(++--+-+---+--+++++---++++--+-)$ $C=(++-+-+++++--++--+++-+------+)$ $D=(++++++-+++---+-++++-++-+-+-+)$
59	$8^2+6^2+3^2+3^2$	$A=(+++++-++++-+--++--++++-+---+-+)$ $B=(-++++-++++-+--++--++++-+---+-+)$ $C=(+++-+---+-++---+--++++-++++--)$ $D=(+-++++-++++--+---++-+---+-++-)$
61	$5^2+4^2+4^2+2^2$	$A=(+-+-+-+-+--+--+--++++++-++---++)$ $B=(+---------+++---++-+-+---++-++-)$ $C=(++-++-----+--+-+----+--++---+-)$ $D=(+++-+--+-++----++-----+++-+++-)$

Table 2 (cont.): Base sequences BS(n): $(A, B; C, D)$ with lengths
$m+1, m+1, m, m,$
where $n = 2m+1$ and $2n = a^2+b^2+c^2+d^2$.

4 Multiplication of sequences and the sum of squares

Koukouvinos and Seberry [11] reformulated some results of Yang [14] and explicitly gave a method to multiply certain sequences by y to get four (disjoint) T-sequences of lengths $y(2m+1)$. We refer to a positive (preferably odd) integer, y, as a Yang number if it can be used to multiply certain sequences with desired properties to get new, longer, seqences with the required desirable properties. So base sequences of lengths $m+1$, $m+1$, m, m can be multiplied by y to get sequences of length $y(2m+1)$ which are T-sequences. Koukouvinos [6] shows $y = 61$ is a Yang number. Combining all known results we have:

(i) Yang numbers exist for $y \in \{n : n \leq 33,\ n = 37, 39, 41, 45, 49, 51, 53, 57, 59, 61, 65, 81, \ldots,$ and $n = 2g+1,\ g = 2^a 10^b 26^c,\ a, b, c$ are non-negative integers$\}$;

(ii) Base sequences of lengths $m+1, m+1, m, m$ ie. BS$(2m+1)$ exist for $m \in \{1, 2, \ldots, 30\} \cup G$, where $G = \{g : g = 2^a 10^b 26^c,\ a, b, c$ non-negative integers$\}$;

(iii) Base sequences of lengths $2n-1, 2n-1, n, n$ exist for $n \in \{2, 4, 6, \ldots, 20, 22, 24\}$.

Hence using the result that, if there are base sequences of lengths $m+p$, $m+p$, m, m and y is a Yang number there are T-sequences of lengths $(2m+p)y$, we have that T-sequences exist for

$$O_1 = \{t : t \text{ odd } \leq 65\} \cup \{71\} \cup \{s : 75 \leq s \leq 199,\ s \text{ not prime }\}$$
$$O_2 = \{y(2m+1) : y \text{ a Yang number},\ m \text{ a base sequence}\}$$

$$\begin{aligned} O_3 &= \{g + g' : g, g' \in G \text{ (eg. we may take } g' = 1)\} \\ E_1 &= \{2yp : y \text{ a Yang number and } p \in O_1 \cup O_2 \cup O_3 \cup E_1\}. \end{aligned}$$

The results that we now give arise from composition theorems due to C.H. Yang [14] which are reformulated by Koukouvinos and Seberry [11] and summarized in Theorem 3.

Definition 2 $A = \{a_1, ..., a_{m+1}\}$, $B = \{b_1, ..., b_{m+1}\}$ and $C = \{c_1, ..., c_m\}$, $D = \{d_1, ..., d_m\}$ are are said to be *suitable sequences* of length $m+1, m+1, m, m$ if they have zero periodic or non-periodic autocorrelation function (as appropriate) and $a_j \neq 0 \Rightarrow b_j = 0$, $b_j \neq 0 \Rightarrow a_j = 0$, $c_j \neq 0 \Rightarrow d_j = 0$, and $d_j \neq 0 \Rightarrow c_j = 0$.

Theorem 3 *There exist four (disjoint) T-sequences of lengths $y(2m+1)$ corresponding to decompositions indicated in Table 3, for $y = 13, 31, 37, 39, 41, 49, 53, 57, 61$.*

Proof. Let a, b, c, d be the row sums of suitable sequences of length $m+1, m+1, m, m$ so that $2m+1 = a^2 + b^2 + c^2 + d^2$. Then using Yang's method to multiply by y we get four (disjoint) T-sequences of lengths $y(2m+1)$ corresponding to a decomposition indicated in Table 3, for $y = 13, 31, 37, 39, 41, 49, 53, 57, 61$.

These kinds of results which specify decompositions into squares have been extensively used [10, 7, 12], to calculate the excess of Hadamard matrices and find regular symmetric Hadamard matrices of order $4k^2$ and SBIBD$(4k^2, 2k^2 \pm k, k^2 \pm k)$ for many values of k (see Seberry and Yamada [13]).

Remark 2 We note that by permuting the variables (or making them positive or negative) may give different sums of squares as

$$\begin{aligned} & (x^2+y^2+z^2+w^2)(2s+1) = (x^2+y^2+z^2+w^2)(a^2+b^2+c^2+d^2) \\ = \; & (ax+by+cz+dw)^2 + (bx-ay+dz-cw)^2 + (cx-dy-az+bw)^2 + (dx+cy-bz-aw)^2 \\ = \; & (bx+ay+dz+cw)^2 + (ax-by+cz-dw)^2 + (dx-cy-bz+aw)^2 + (cx+dy-az-bw)^2 \\ = \; & (cx+dy+az+bw)^2 + (dx-cy+bz-aw)^2 + (ax-by-cz+dw)^2 + (bx+ay-dz-cw)^2 \\ = \; & (dx+cy+bz+aw)^2 + (cx-dy+az-bw)^2 + (bx-ay-dz+cw)^2 + (ax+by-cz-dw)^2. \end{aligned}$$

Remark 3 We note that choosing a, b, c and d in Table 3 for the decomposition $39 = 5^2 + 3^2 + 2^2 + 1^2$ we get for different permutations different sums of squares:

(a, b, c, d)	$13 \cdot 39$	$(3a+2b)^2 + (2a-3b)^2 + (3c+2d)^2 + (2c-3d)^2$
5, 3, 2, 1	$13 \cdot 39$	$= 21^2 + 1^2 + 8^2 + 1^2$
5, 3, 1, 2	$13 \cdot 39$	$= 21^2 + 1^2 + 7^2 + 4^2$
3, 5, 2, 1	$13 \cdot 39$	$= 19^2 + 9^2 + 8^2 + 1^2$
3, 5, 1, 2	$13 \cdot 39$	$= 19^2 + 9^2 + 7^2 + 4^2$
5, 2, 3, 1	$13 \cdot 39$	$= 19^2 + 4^2 + 11^2 + 3^2$
2, 5, 3, 1	$13 \cdot 39$	$= 16^2 + 11^2 + 11^2 + 3^2$
2, 5, 3, 1	$13 \cdot 39$	$= 16^2 + 11^2 + 9^2 + 7^2$
5, 1, 3, 2	$13 \cdot 39$	$= 17^2 + 7^2 + 13^2 + 0^2$
5, 1, 2, 3	$13 \cdot 39$	$= 17^2 + 7^2 + 12^2 + 5^2$
1, 5, 3, 2	$13 \cdot 39$	$= 13^2 + 13^2 + 13^2 + 0^2$
1, 5, 2, 3	$13 \cdot 39$	$= 13^2 + 13^2 + 12^2 + 5^2$

$y = 13$ $13(2s+1)$	$=$	$(3d-2a)^2 + (3c+2b)^2 + (3b-2c)^2 + (3a+2d)^2$
$y = 31$ $31(2s+1)$	$=$	$(a-b+5c+2d)^2 + (a+b-2c+5d)^2 + (5a-2b-c-d)^2$
$y = 37$ $37(2s+1)$	$=$	$(4a-b+4c+2d)^2 + (a+4b-2c+4d)^2 + (4a-2b-4c-d)^2$
$y = 39$ $39(2s+1)$	$=$	$(5a-b+3c+2d)^2 + (a+5b-2c+3d)^2 + (3a-2b-5c-d)^2$
$y = 41$ $41(2s+1)$	$=$	$(2a-b+6c)^2 + (a+2b+6d)^2 + (6a-2c-d)^2 + (6b+c-2d)^2$
$y = 49$ $49(2s+1)$	$=$	$(-4a+2b-2c-5d)^2 + (-2a-4b+5c-2d)^2 + (2a-5b-4c-2d)^2$ $+(5a+2b+2c-4d)^2$
$y = 53$ $53(2s+1)$	$=$	$(6a-c+4d)^2 + (6b-4c-d)^2 + (a+4b+6c)^2 + (-4a+b+6d)^2$
$y = 57$ $57(2s+1)$	$=$	$(6a+4b-2c-d)^2 + (-4a+6b+c-2d)^2 + (2a-b+6c-4d)^2$ $+(a+2b+4c+6d)^2$
$y = 61$ $61(2s+1)$	$=$	$(4a+4b-2c-5d)^2 + (4a-4b-5c+2d)^2 + (-2a+5b-4c+4d)^2$ $+(5a+2b+4c+4d)^2$

Table 3: Decompositions arising from Yang's composition $y(2s+1)$ where $2s+1 = a^2+b^2+c^2+d^2$.

References

[1] G. Cohen, D. Rubie, J. Seberry, C. Koukouvinos, S. Kounias, and M. Yamada. A survey of base sequences, disjoint complementary sequences and $OD(4t; t, t, t, t)$. *JCMCC*, 5:69-104, 1989.

[2] Genet M. Edmondson, Jennifer Seberry and Malcolm R. Anderson. On the existence of Turyn sequences of length less than 43, *Math. Comput.*, 62:351-362, 1994.

[3] S. Eliahou, M. Kervaire, and B. Saffari. A new restriction on the lengths of Golay complementary sequences. *J. Combin. Th. (Ser A)*, 55:49-59, 1990.

[4] Marc Gysin, *On Normal and Near-Yang Sequences*, M.Sc. Thesis, University of Wollongong, 1993.

[5] Marc Gysin and Jennifer Seberry, New results with near-Yang sequences, *Bull. Inst. Comb. Appl.*, to appear.

[6] C. Koukouvinos, Construction of some sequences with zero autocorrelation function, *JCMCC*, to appear.

[7] Christos Koukouvinos, Stratis Kounias, and Jennifer Seberry, Further Hadamard matrices with maximal excess and new SBIBD($4k^2$, $2k^2+k$, k^2+k), *Utilitas Math.*, 36:135-150, 1989.

[8] C. Koukouvinos, S. Kounias, and K. Sotirakoglou. On base and Turyn sequences. *Math. Comput.*, 55:825-837, 1990.

[9] C. Koukouvinos, S. Kounias, J. Seberry, C.H. Yang and J. Yang. On sequences with zero autocorrelation, *Codes, Designs and Crypt*, to appear.

[10] Christos Koukouvinos and Jennifer Seberry, Construction of new Hadamard matrices with maximal excess and infinitely many new SBIBD($4k^2$, $2k^2+k$, k^2+k), *Graphs, Matrices and Designs: A Festschrift for Norman J. Pullman*, by Rolf Rees, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, 1992.

[11] C. Koukouvinos and J. Seberry. Addendum to Further results on base sequences, disjoint complementary sequences, OD($4t; t, t, t, t$) and the excess of Hadamard matrices. *Congr. Num.*, 82:97-103, 1991.

[12] Jennifer Seberry, Existence of SBIBD($4k^2$, $2k^2+k$, k^2+k) and Hadamard matrices with maximal excess, *Austral. J. Combinatorics*, 4:87-92, 1991.

[13] Jennifer Seberry and Mieko Yamada, Hadamard matrices, Sequences, and Block Designs, *Contemporary Design Theory - A Collection of Surveys*, Eds D. J. Stinson and J. Dinitz, John Wiley and Sons, 431-560, 1992.

[14] C.H. Yang. On composition of four-symbol δ-codes and Hadamard matrices. *Proc. Amer. Math. Soc.*, 107:763-776, 1989.

[15] C.H. Yang. On Golay, near normal and base sequences. In preparation.

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