Edge-Neighbor-Integrity of Trees

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Abstract. The edge-neighbor-integrity of a graph G, ENI(G), is defined to be $\mathrm{ENI(G)} = \min_{S \subseteq E(G)} \{|S| + \omega(G/S)\}$, where S is any edge subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S. In this paper, we find the minimum and maximum edge-neighbor-integrity among all trees with any fixed order, and also show that for any integer l between the extreme values there is a tree with the edge-neighbor-integrity l.

I. Introduction

In 1987 Barefoot, Entringer, and Swart introduced the integrity of a graph to measure the "vulnerability" of the graph. [1,2] In 1994 [4] we developed a graph parameter, called "vertex-neighbor-integrity", incorporating the concept of the integrity [1,2] and the idea of the vertex-neighbor-connectivity [5]. Here we consider the edge- analogue of vertex-neighbor-integrity, incorporating the concept of the integrity and the idea of the edge-neighbor-connectivity [3].

Let G = (V,E) be a graph. An edge $e = [u, v]$ in G is said to be *subverted* when the incident vertices, u, v, of the edge e are deleted from G. A set of edges $S = \{e_1, e_2, ..., e_m\}$ is called an *edge subversion strategy* of G if each of the edges in S has been subverted from G. Let G/S be the survival-subgraph left when S has been an edge subversion strategy of G. The *edge-neighbor-integrity* of a graph G, ENI(G), is defined to be

$$\mathrm{ENI(G)} = \min_{S \subseteq E(G)} \{|S| + \omega(G/S)\},$$

where S is any edge subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S.

Example: $K_{n,m}$, where $n > 1$ and $m > 1$, is a complete bipartite graph with a bipartition (X,Y), where $|X| = n$ and $|Y| = m$.

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$$\mathrm{ENI}(K_{n,m}) = \min_{S \subseteq E(K_{n,m})} \{|S| + \omega(K_{n,m}/S)\}$$

$= |S^*| + \omega(K_{n,m}/S^*)$, where S^* is a set of matching saturating each vertex of X if $|X| \leq |Y|$ (or Y if $|Y| \leq |X|$),

$$= \begin{cases} n+1, & \text{if } n < m; \\ m+1, & \text{if } m < n; \\ m \text{ or } n, & \text{if } m = n. \end{cases}$$

In this paper, we find the minimum and maximum edge-neighbor-integrity among all trees with any fixed order, and also show that for any integer l between the extreme values there is a tree whose edge-neighbor-integrity is l. $\lceil x \rceil$ is the smallest integer greater than or equal to x. $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

II. The Minimum and Maximum Edge-Neighbor-Integrity of Trees

For any connected graph G of order at least 3, the edge-neighbor-integrity, $\mathrm{ENI}(G) \geq 2$, since there is no edge e in G such that $G/\{e\} = \emptyset$. Trees are connected graphs, and therefore $\mathrm{ENI}(T) \geq 2$, for any tree T of order at least 3. If we can find a tree of order at least 3 whose edge-neighbor-integrity is 2, then the minimum edge-neighbor-integrity among all trees is 2.

Lemma 1: Let G be a connected graph of order at least 3. If $\mathrm{ENI}(G) = 2$, then the diameter of G is ≤ 3.

Proof: Assume that the diameter of G is ≥ 4, then G contains a path P_5. Hence for any edge e in G, $\omega(G/\{e\}) \geq 2$, and for any two edges e_1 and e_2 in G, $\omega(G/\{e_1, e_2\}) \geq 1$. Therefore $\mathrm{ENI}(G) \geq 3$, a contradiction. Hence the diameter of G is ≤ 3. QED.

Let $K_{1,n}$ be a complete bipartite graph with a vertex bipartition (X,Y), where $|X| = 1$ and $|Y| = n$. We also call $K_{1,n}$ a star with $n+1$ vertices. Let $\mathrm{DS}(n_1, n_2)$ be a double star with $\{n_1, n_2\}$ end-vertices, where $n_1 \geq 0$ and $n_2 \geq 0$, and a common edge $[u, v]$, as shown in Figure 1. Note that if either n_1 or n_2 is 0, then the double star $\mathrm{DS}(n_1, n_2)$ is a star.

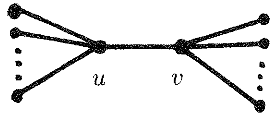
$DS(n_1, n_2)$: n_1 vertices  n_2 vertices

Figure 1

Then we have the following theorem.

Theorem 2: Let T be a tree of order $n \geq 3$. Then ENI(T) = 2 if and only if T is either a star $K_{1,n-1}$ or a double star $DS(n_1, n_2)$, where $n_1 \geq 1$, $n_2 \geq 1$, and $n_1 + n_2 = n - 2$.

Proof: If T is a tree of order at least 3 and ENI(T) = 2, then by Lemma 1, the diameter of T is either 2 or 3. If the diameter of T is 2, then T is a star $K_{1,n-1}$. If the diameter of T is 3, then T is a double star $DS(n_1, n_2)$, where $n_1 \geq 1$, $n_2 \geq 1$, and $n_1 + n_2 = n - 2$.

Conversely, let T be either a star $K_{1,n-1}$ with the order $n \geq 3$ or a double star $DS(n_1, n_2)$, where $n_1 \geq 1$, $n_2 \geq 1$, and the order $n = n_1 + n_2 + 2 \geq 4$. Then the subversion of any one edge e from $K_{1,n-1}$ produces $n - 2$ isolated vertices. Hence

$$\mathrm{ENI}(K_{1,n-1}) = \min_{S \subseteq E(G)} \{|S| + \omega(G/S)\}$$

$$= |\{e\}| + \omega(G/\{e\}) = 1 + 1 = 2.$$

The subversion of the common edge e from $DS(n_1, n_2)$ produces $n_1 + n_2$ isolated vertices; the subversion of any another edge from $DS(n_1, n_2)$ produces a subgraph with the maximum order of the components ≥ 2. Hence

$$\mathrm{ENI}(DS(n_1, n_2)) = \min_{S \subseteq E(G)} \{|S| + \omega(G/S)\}$$

$$= |\{e\}| + \omega(G/\{e\}) = 1 + 1 = 2.$$

QED.

Since $DS(0, n-2)$ ($=K_{1,n-1}$), $DS(1, n-3)$, $DS(2, n-4)$, ..., and $DS(\lfloor n/2 \rfloor - 1, n - \lfloor n/2 \rfloor - 1)$ are all of the trees with the order n, where $n \geq 3$, and the edge-neighbor-integrity is 2, there are $\lfloor n/2 \rfloor$ non-isomorphic trees of order n with the minimum edge-neighbor-integrity.

Next, we find the maximum edge-neighbor-integrity among all trees of order $n \geq 1$.

Lemma 3: For positive integers, n and m, if n is fixed, then the function $g(m) = m + \lceil n/m \rceil$ has the minimum value $\lceil 2\sqrt{n} \rceil$ at $m = \lceil \sqrt{n} \rceil$. [2]

Theorem 4: Let P_n be a path of order $n \geq 1$. Then

$$\text{ENI}(P_n) = \lceil 2\sqrt{n+2} \rceil - 3.$$

Proof: Let $V(P_n)=\{v_1, v_2, v_3, ..., v_n\}$ and S be any subset of $E(P_n)$. The subversion of an edge $e = [v_i, v_{i+1}]$ from P_n is the removal of the vertices v_i and v_{i+1} from P_n, so

$$\omega(P_n/S) \geq \left\lceil \frac{(n-2|S|)}{(|S|+1)} \right\rceil.$$

Let $|S| = m$.

$$\text{ENI}(P_n) = \min_{S \subseteq E(P_n)} \{|S| + \omega(P_n/S)\}$$

$$\geq \min_{m \geq 0} \left\{ m + \left\lceil \frac{n-2m}{m+1} \right\rceil \right\} \qquad (1)$$

$$= -3 + \min_{m \geq 0} \left\{ m+1+ \left\lceil \frac{n+2}{m+1} \right\rceil \right\}$$

$$= -3 + \lceil 2\sqrt{n+2} \rceil. \qquad \text{(By Lemma 3.)}$$

Setting $|S| = m = \lceil \sqrt{n+2} \rceil - 1$ gives the minimum value of $\{m + \lceil (n-2m)/(m+1) \rceil\}$ and the equality of (1) holds by taking S to be a set of m edges with equal distance in P_n. $m = \lceil \sqrt{n+2} \rceil - 1$ and $n - 2m \geq 0$ if and only if $n \geq 2$ and $n \neq 3$. Therefore, if $n \geq 2$ and $n \neq 3$, then the set S is taken to be a set of $\lceil \sqrt{n+2} \rceil - 1$ edges with equal distance in P_n. If $n = 1$, then $\text{ENI}(P_n)=1$ and $\lceil 2\sqrt{n+2} \rceil - 3 = 1$. If $n = 3$, then $\text{ENI}(P_n)=2$ and $\lceil 2\sqrt{n+2} \rceil - 3 = 2$. Hence we obtain the result. QED.

To show that a path P_n has the maximum edge-neighbor-integrity among all trees of order n, we first show the following theorem.

Theorem 5: If T is a tree of order n and $0 \leq m \leq n-1$, then there is a subset S $\subseteq E(T)$ such that $|S| = m$ and $\omega(T/S) \leq \lceil (n-2m)/(m+1) \rceil$.

Proof: Assume that the result is not true for some n, and let T be a tree of order n with largest diameter, say d, satisfying

$$\omega(\mathrm{T/S}) > \left\lceil \frac{(n-2|\mathrm{S}|)}{(|\mathrm{S}|+1)} \right\rceil,$$

for any subset $\mathrm{S} \subseteq \mathrm{E(T)}$. From the proof of Theorem 4, we know that $\mathrm{T} \ncong \mathrm{P}_n$, i.e., $d \leq n-2$. Let $\mathrm{P}=(v_1, v_2, ..., v_{d+1})$ be a longest path in T. Then there is a vertex v in the path P such that the degree of v is greater than 2; let the least index of such vertices be k. Then $1 < k < d+1$. Now construct the tree T' which is T $-[v_k, v_{k+1}] + [v_1, v_{k+1}]$ (as shown in Figure 2).

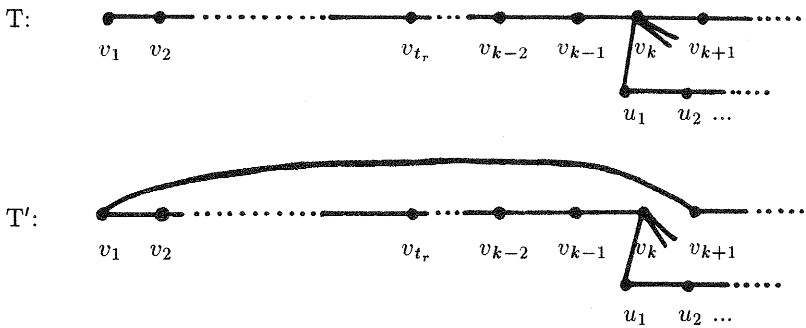


Figure 2

Since the order of T' is n and diameter $d' > d$, by the assumption on T, there is an edge-subset $\mathrm{S}' \subseteq \mathrm{E(T')}$ such that $|\mathrm{S}'| = m$ and

$$\omega(\mathrm{T'/S'}) \leq \left\lceil \frac{(n-2m)}{(m+1)} \right\rceil.$$

Obviously, $\mathrm{T}/\{e\} \cong \mathrm{T}'/\{e\}$ if the edge e is incident with v_{k+1} in T' and $e \neq [v_1, v_{k+1}]$, and $\mathrm{T}/\{f\} \subseteq \mathrm{T}'/\{f\}$ if the edge f is incident with v_k in T'. It follows that $e, f \notin \mathrm{S}'$, for all edges e incident with v_{k+1} in T' and $e \neq [v_1, v_{k+1}]$, and for all edges f incident with v_k in T', since otherwise taking $\mathrm{S} = \mathrm{S}'$ gives $\omega(\mathrm{T/S}) \leq \omega(\mathrm{T'/S'}) \leq \lceil (n-2m)/(m+1) \rceil$, a contradiction.

Next, we show that $[v_1, v_{k+1}] \notin \mathrm{S}'$.

Assume that $[v_1, v_{k+1}] \in \mathrm{S}'$. If the edge $[v_1, v_2] \in \mathrm{S}'$, then let S be S' with $[v_1, v_{k+1}]$ replaced by $[v_k, v_{k+1}]$. Then $\mathrm{T/S} \subseteq \mathrm{T'/S'}$ and $\omega(\mathrm{T/S}) \leq \omega(\mathrm{T'/S'}) \leq \lceil (n-2m)/(m+1) \rceil$, a contradiction. If there are edges $[v_{t_1}, v_{t_1+1}]$, ..., $[v_{t_r}, v_{t_r+1}]$, where $2 \leq t_1 < t_2 < ... < t_r \leq k-2$, in S', then let S be S' with $[v_{t_i}, v_{t_i+1}]$ replaced by $[v_{t_i-1}, v_{t_i}]$, for all $t_1, t_2, ..., t_r$, and $[v_1, v_{k+1}]$ replaced by $[v_k, v_{k+1}]$, then T/S and $\mathrm{T'/S'}$ have different components as follows:

T/S has the components ——

path $\mathcal{P}_0 = (v_1, ..., v_{t_1-2})$, only if $t_1 \geq 3$,
path $\mathcal{P}_j = (v_{t_j+1}, ..., v_{t_{j+1}-2})$, where $1 \leq j \leq r-1$,
path $\mathcal{P}_r = (v_{t_r+1}, ..., v_{k-1})$,
$\mathcal{C}_k$: the component containing u_i $(i = 1, 2, ...)$
(as shown in Figure 2).

T′/S′ has the components ——

path $\mathcal{P}'_0 = (v_2, ..., v_{t_1-1})$, only if $t_1 \geq 3$,
path $\mathcal{P}'_j = (v_{t_j+2}, ..., v_{t_{j+1}-1})$, where $1 \leq j \leq r-1$,
$\mathcal{C}'_r$: the component containing a $(k - t_r - 1)$-path — $(v_{t_r+2}, ..., v_k)$, and containing u_i $(i = 1, 2, ...)$
(as shown in Figure 2).

Other than the above, T/S and T′/S′ have the same components. The order of $\mathcal{P}_0$ = the order of $\mathcal{P}'_0$, the order of $\mathcal{P}_j$ = the order of $\mathcal{P}'_j$, for all $1 \leq j \leq r-1$, the order of $\mathcal{P}_r <$ the order of $\mathcal{C}'_r$, and the order of $\mathcal{C}_k \leq$ the order of $\mathcal{C}'_r$, hence all of the components of T/S have sizes smaller than or equal to $\omega(\mathrm{T}'/\mathrm{S}')$, and $\omega(\mathrm{T/S}) \leq \omega(\mathrm{T}'/\mathrm{S}') \leq \lceil (n-2m)/(m+1) \rceil$, a contradiction.

Therefore $[v_1, v_{k+1}] \notin \mathrm{S}'$.

It has been shown that e, $f \notin \mathrm{S}'$, where e is incident with v_{k+1} in T′, and f is incident with v_k in T′, hence v_k and v_{k+1} must be in T′/S′. It follows that there must exist $v_{i_1}, v_{i_2}, ..., v_{i_r}$, $(r \geq 1)$, where $1 \leq i_1 < i_2 < ... < i_r \leq k-2$, such that $e_{i_1} = [v_{i_1}, v_{i_1+1}]$, $e_{i_2} = [v_{i_2}, v_{i_2+1}]$, ..., $e_{i_r} = [v_{i_r}, v_{i_r+1}] \in \mathrm{S}'$, since otherwise v_k and v_{k+1} are in the same component of T′/S′, thus taking S=S′ gives $\omega(\mathrm{T/S}) = \omega(\mathrm{T}'/\mathrm{S}') \leq \lceil (n-2m)/(m+1) \rceil$, a contradiction.

Let S* be S′ with $[v_{i_j}, v_{i_j+1}]$ replaced by $[v_{i_j+k-i_r}, v_{i_j+k-i_r+1}]$, for all $1 \leq j \leq r$. Since $i_r \leq k-2$, $3 \leq i_1 + k - i_r < i_2 + k - i_r < i_3 + k - i_r < ... < i_r + k - i_r = k$. By the assumption on T, $\omega(\mathrm{T/S}^*) > \lceil (n-2m)/(m+1) \rceil$, and all of the components of T/S*, except the path $\mathrm{P}^* = (v_1, v_2, ..., v_{i_1+k-i_r-1})$, have the sizes smaller than or equal to $\omega(\mathrm{T}'/\mathrm{S}')$, which is $\leq \lceil (n-2m)/(m+1) \rceil$, hence the order of P* must be

$$i_1 + k - i_r - 1 \geq \left\lceil \frac{n-2m}{m+1} \right\rceil + 1.$$

Let $\mathcal{A}'_k$ and $\mathcal{A}'_{k+1}$ be two different components of T′/S′ containing v_k and v_{k+1}, respectively, and h be the number of the vertices in $\mathcal{A}'_{k+1}$ that are not in the set $\{v_1, v_2, ..., v_{i_1-1}\}$. Since the order of $\mathcal{A}'_{k+1}$ is less than or equal to $\lceil (n-2m)/(m+1) \rceil$, we have

$$1 \leq h \leq \left\lceil \frac{n-2m}{m+1} \right\rceil - (i_1 - 1) \leq k - i_r - 1.$$

Now, let S be the set S′ with $[v_{i_j}, v_{i_j+1}]$ replaced by $[v_{i_j+h}, v_{i_j+h+1}]$, for all $1 \le j \le r$, and consider the sizes of the components of T/S. By the constructions of S and S′, all of the components of T/S, except those containing v_1 and v_k, have at most $\lceil (n-2m)/(m+1) \rceil$ vertices. The vertex set of the component of T/S containing v_1 is obtained from the vertex set of $\mathcal{A}'_{k+1}$ by deleting the h vertices $\mathcal{A}'_{k+1} - \{v_1, v_2, \ldots, v_{i_1-1}\}$ and appending the vertices $v_{i_1}, v_{i_1+1}, \ldots, v_{i_1+h-1}$ with no change in number of vertices. Similarly, the vertex set of the component of T/S containing v_k is obtained from the vertex set of $\mathcal{A}'_k$ by deleting the h vertices $v_{i_r+2}, v_{i_r+3}, \ldots, v_{i_r+h}, v_{i_r+h+1}$ and appending the h vertices, $\mathcal{A}'_{k+1} - \{v_1, v_2, \ldots, v_{i_1-1}\}$ with no change in number of vertices. Hence $\omega(\mathrm{T/S}) \le \lceil (n-2m)/(m+1) \rceil$, a contradiction.

Therefore we obtain the result of the theorem. QED.

Using Theorem 5, we now show that the path P_n has the maximum edge-neighbor-integrity among all trees of order n.

Theorem 6: The path P_n has the maximum edge-neighbor-integrity among all trees of order $n \ge 1$.

Proof: It is trivial for $n = 1$.

Let T be a tree of order $n \ge 2$. Then by Theorem 5, for any integer m, $0 \le m \le n-1$, there is an edge-subset $S' \subseteq E(T)$ such that $|S'| = m$ and $\omega(T/S') \le \lceil (n-2m)/(m+1) \rceil$.

$$\mathrm{ENI(T)} = \min_{S \subseteq E(T)} \{|S| + \omega(T/S)\}$$

$$\le \min_{0 \le m \le n-1} \left\{ m + \left\lceil \frac{n-2m}{m+1} \right\rceil \right\}.$$

By the proof of Theorem 4, $\mathrm{ENI}(P_n) = m + \lceil (n-2m)/(m+1) \rceil$ with $m = \lceil \sqrt{n+2} \rceil - 1$. $0 \le \lceil \sqrt{n+2} \rceil - 1 \le n-1$ if and only if $n \ge 2$. Therefore

$$\mathrm{ENI(T)} \le \min_{0 \le m \le n-1} \left\{ m + \left\lceil \frac{n-2m}{m+1} \right\rceil \right\}$$

$$\le m^* + \left\lceil \frac{n-2m^*}{m^*+1} \right\rceil, \quad \text{where } m^* = \lceil \sqrt{n+2} \rceil - 1$$

$$= \mathrm{ENI}(P_n).$$

QED.

We have shown that the path P_n has the maximum edge-neighbor-integrity among all trees of order n. However, P_n is not the only tree that has the maximum edge-neighbor-integrity. We evaluate the edge-neighbor-integrity of $T_{n,k}$ (as shown in Figure 3), where $1 \leq k \leq n-2$, in Theorem 8, stating that there are at least $\lfloor \sqrt{n+2} - (9/4) \rfloor$ non-isomorphic trees of order n having the same edge-neighbor-integrity as P_n.

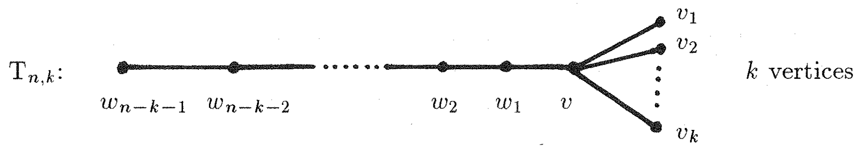


Figure 3

Lemma 7: There is a unique path P_n satisfying the following condition (A) —— for any subset S of $E(P_n)$, if $ENI(P_n) = |S| + \omega(P_n/S)$ then $\omega(P_n/S) = 0$. Moreover, $n = 2$.

Proof: Let P_n satisfy the condition (A). By the proof of Theorem 4, if $n \geq 2$ and $n \neq 3$, then there is an edge subset S^* of $E(P_n)$ such that $ENI(P_n) = |S^*| + \omega(P_n/S^*)$, where

$$\omega(P_n/S^*) = \left\lceil \frac{n - 2|S^*|}{|S^*| + 1} \right\rceil$$

and

$$|S^*| = \lceil \sqrt{n+2} \rceil - 1.$$

Since P_n satisfies the condition (A),

$$\begin{aligned} ENI(P_n) &= |S^*| + \omega(P_n/S^*) \\ &= |S^*| \\ &= \lceil \sqrt{n+2} \rceil - 1. \end{aligned}$$

By Theorem 4,

$$ENI(P_n) = \lceil 2\sqrt{n+2} \rceil - 3.$$

Therefore

$$\lceil \sqrt{n+2} \rceil - 1 = \lceil 2\sqrt{n+2} \rceil - 3,$$

and hence $n = 2$ or 4.

Let $P_4 = (v_1, v_2, v_3, v_4)$. Then $S_1 = \{[v_1, v_2], [v_3, v_4]\}$ and $S_2 = \{[v_2, v_3]\}$ satisfy

$$\begin{aligned} ENI(P_4) &= |S_1| + \omega(P_4/S_1) \\ &= |S_2| + \omega(P_4/S_2) \\ &= 2. \end{aligned}$$

$\omega(P_4/S_1) = 0$, but $\omega(P_4/S_2) = 1 \neq 0$. Therefore the path P_4 does not satisfy the condition (A).

Let $P_2 = (v_1, v_2)$. $S = \{[v_1, v_2]\}$ is the only edge subset of $E(P_2)$ satisfying $ENI(P_2) = |S| + \omega(P_2/S) = 1$, and $\omega(P_2/S) = 0$.

The remaining case is that $n = 3$: Let $P_3 = (v_1, v_2, v_3)$. Then $S_1 = \{[v_1, v_2], [v_2, v_3]\}$ and $S_2 = \{[v_1, v_2]\}$ satisfy

$$\begin{aligned} ENI(P_3) &= |S_1| + \omega(P_3/S_1) \\ &= |S_2| + \omega(P_3/S_2) \\ &= 2. \end{aligned}$$

$\omega(P_3/S_1) = 0$, but $\omega(P_3/S_2) = 1 \neq 0$. Therefore the path P_3 does not satisfy the condition (A).

Hence P_2 is the only path satisfying the condition (A). QED.

Theorem 8: The edge-neighbor-integrity of $T_{n,k}$ (as shown in Figure 3), where $n \geq 3$ and $1 \leq k \leq n-2$, is as follows:

$$ENI(T_{n,k}) = \begin{cases} \lceil 2\sqrt{n+2} \rceil - 3, & \text{if } 1 \leq k \leq \sqrt{n+2} - \frac{9}{4}; \\ \lceil 2\sqrt{n-k} \rceil - 2, & \text{if } \sqrt{n+2} - \frac{9}{4} \leq k \leq n-5; \\ 3, & \text{if } k = n-4; \\ 2, & \text{if } k = n-3, n-2. \end{cases}$$

Proof: If $k = n-2$, $T_{n,k}$ is a star. Then $ENI(T_{n,k}) = 2$.

If $k = n-3$, $T_{n,k}$ is a double star. Then $ENI(T_{n,k}) = 2$.

Now we consider the case of $k \leq n-4$. Let S^* be a subset of $E(T_{n,k})$ for which $ENI(T_{n,k}) = |S^*| + \omega(T_{n,k}/S^*)$.

If $[v, v_i] \in S^*$, for some i, $1 \leq i \leq k$, we may let S' be S^* with $[v, v_i]$ replaced by $[w_1, v]$. Then

$$\begin{aligned} |S'| + \omega(T_{n,k}/S') &\leq |S^*| + \omega(T_{n,k}/S^*) \\ &= ENI(T_{n,k}) \\ &= \min_{S \subseteq E(T_{n,k})} \left\{ |S| + \omega(T_{n,k}/S) \right\}. \end{aligned}$$

Therefore

$$ENI(T_{n,k}) = |S'| + \omega(T_{n,k}/S').$$

Hence without loss of generality we may assume that $[v, v_i] \notin S^*$, for all $1 \le i \le k$.
Now we consider two cases:

Case 1. If $[w_1, v] \in S^*$, then

$$\mathrm{ENI}(T_{n,k}) = \begin{cases} \mathrm{ENI}(P_{n-(k+2)}) + 1, & \text{if } n-(k+2) \neq 2; \\ \mathrm{ENI}(P_{n-(k+2)}) + 2, & \text{if } n-(k+2) = 2. \end{cases}$$

(By Lemma 7.)

$$= \begin{cases} \lceil 2\sqrt{n-k} \rceil - 2, & \text{if } k \neq n-4; \\ 3, & \text{if } k = n-4. \end{cases}$$

(By Theorem 4.)

Case 2. If $[w_1, v] \notin S^*$, then $v, v_1, v_2, \ldots,$ and v_k are in the same component of $T_{n,k}/S^*$, and

$$\mathrm{ENI}(T_{n,k}) = \mathrm{ENI}(P_n) = \lceil 2\sqrt{n+2} \rceil - 3.$$

Hence,

$$\mathrm{ENI}(T_{n,k}) = \begin{cases} \min\limits_{k \neq n-4} (\lceil 2\sqrt{n-k} \rceil - 2, \lceil 2\sqrt{n+2} \rceil - 3) & \text{or} \\ \min\limits_{k = n-4} (3, \lceil 2\sqrt{n+2} \rceil - 3). \end{cases}$$

In the case of $k = n-4$, $\lceil 2\sqrt{n+2} \rceil - 3 \le 3$ if and only if $n \le 7$. If $n \le 7$, $k \ge 1$, and $k = n-4$, then n can only be 7, 6, or 5. When $n = 7$, 6, or 5, $\lceil 2\sqrt{n+2} \rceil - 3 = 3$. Hence, in the case of $k = n-4$, $\mathrm{ENI}(T_{n,k}) = 3$.

In the case of $k \neq n-4$, $\lceil 2\sqrt{n+2} \rceil - 3 \le \lceil 2\sqrt{n-k} \rceil - 2$ if $k \le \sqrt{n+2} - (9/4)$, and $\lceil 2\sqrt{n-k} \rceil - 2 \le \lceil 2\sqrt{n+2} \rceil - 3$ if $k \ge \sqrt{n+2} - (9/4)$.

Therefore,

$$\mathrm{ENI}(T_{n,k}) = \begin{cases} \lceil 2\sqrt{n+2} \rceil - 3, & \text{if } 1 \le k \le \sqrt{n+2} - \frac{9}{4}; \\ \lceil 2\sqrt{n-k} \rceil - 2, & \text{if } \sqrt{n+2} - \frac{9}{4} \le k \le n-5; \\ 3, & \text{if } k = n-4; \\ 2, & \text{if } k = n-3, n-2. \end{cases}$$

QED.

Among all trees of order $n \geq 3$, the maximum edge-neighbor-integrity is $\lceil 2\sqrt{n+2}\rceil - 3$, and the minimum is 2. We can find a tree whose edge-neighbor-integrity is l, for any integer l between the extreme values, as shown below.

Theorem 9: If l is any integer, where $2 \leq l \leq \lceil 2\sqrt{n+2}\rceil - 3$, then there is a tree T of order n such that ENI(T) $= l$.

Proof: If $l = 2$, $T = K_{1,n-1}$ or $T = DS(i, n-i-2)$, where $1 \leq i \leq \lfloor (n-2)/2 \rfloor$; if $l = \lceil 2\sqrt{n+2}\rceil - 3$, $T = P_n$ or $T = T_{n,k}$, where $1 \leq k \leq \sqrt{n+2} - (9/4)$. Therefore we assume that $2 < l < \lceil 2\sqrt{n+2}\rceil - 3$. Since

$$l < \lceil 2\sqrt{n+2}\rceil - 3,$$

we have

$$l + 3 < 2\sqrt{n+2},$$

and

$$n > \frac{l^2}{4} + \frac{3}{2}l + \frac{1}{4}. \tag{2}$$

Let r be the largest integer such that $\lceil 2\sqrt{r+2}\rceil - 3 = l - 1$, so $\lceil 2\sqrt{(r+1)+2}\rceil - 3 = l$. Since

$$l + 3 \geq 2\sqrt{r+3},$$

we have

$$r + 1 \leq \frac{l^2}{4} + \frac{3}{2}l + \frac{1}{4}. \tag{3}$$

Hence combining (2) and (3),

$$n \geq r + 2.$$

We let $k = n - r - 1 \geq 1$, so that $T_{n,k}$ contains a path P_{r+1}. Then

$$\mathrm{ENI}(T_{n,k}) \geq \mathrm{ENI}(P_{r+1}) = \lceil 2\sqrt{(r+1)+2}\rceil - 3 = l.$$

The subversion of the edge $[v, w_1]$ from $T_{n,k}$ produces k isolated vertices and a path P_{r-1}. Hence

$$\begin{aligned} \mathrm{ENI}(T_{n,k}) &\leq 1 + \mathrm{ENI}(P_{r-1}), \qquad \text{if } r - 1 \neq 2 \\ &= 1 + \lceil 2\sqrt{(r-1)+2}\rceil - 3 \\ &= \lceil 2\sqrt{r+1}\rceil - 2 \\ &\leq \lceil 2\sqrt{r+2}\rceil - 2 = l. \end{aligned}$$

Therefore if $r - 1 \neq 2$, $\text{ENI}(\text{T}_{n,k}) = l$.

The remaining part is to show that $r = 3$ is impossible. r is the largest integer such that $\lceil 2\sqrt{r+2} \rceil - 3 = l - 1$. If $r = 3$ then $l = \lceil 2\sqrt{5} \rceil - 2 = 3$. Thus $\lceil 2\sqrt{(r+1)+2} \rceil - 3 = \lceil 2\sqrt{6} \rceil - 3 = 2 = l - 1$, a contradiction to the assumption on r. Hence $r \neq 3$.

Therefore we have found a tree, $\text{T}_{n,k}$, whose edge-neighbor-integrity is l. QED.

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