EXTREMAL DOUBLY EVEN (56,28,12) CODES AND HADAMARD MATRICES OF ORDER 28

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Abstract. In [2] Bussemaker and Tonchev constructed six doubly even $(56, 28, 12)$ codes from two Hadamard matrices of order 28. But two of them were not distinguished. In [11] and [12] we characterized Hadamard matrices of order 28 and there are exactly 487 Hadamard matrices, up to equivalence. In this paper we show that only two of the above 487 matrices produce six doubly even $(56, 28, 12)$ codes and that two of the six codes are equivalent. Therefore there are exactly five $(56, 28, 12)$ codes, up to equivalence, produced by Hadamard matrices of order 28.

1. INTRODUCTION

A Hadamard matrix H of order n is an $n \times n$ matrix of $\pm 1's$ with $HH^t = nI$. It is well known that n is necessarily 1, 2 or a multiple of four. We say that two matrices M_1 and M_2 of the same size are equivalent if there exists a signed permutation g of rows and columns of M_1 with $M_1^g = M_2$. A matrix which is equivalent to a Hadamard matrix is also a Hadamard matrix. An automorphism of H is a signed permutation g of the set of rows and columns such that $H^g = H$. The set of automorphisms forms a group under composition called the automorphism group of H and it is denoted by $Aut(H)$. We say that a set of four rows of H is a Hall set if the submatrix consisting of the four rows is equivalent to the following matrix:

$$\begin{bmatrix} + & + & + & + & J_m & J_m & J_m & J_m \\ + & + & - & - & J_m & J_m & -J_m & -J_m \\ + & - & + & - & J_m & -J_m & J_m & -J_m \\ + & - & - & + & -J_m & J_m & J_m & -J_m \end{bmatrix}, \tag{1.1}$$

where J_m is the all $1's$ row vector of dimension $(n-4)/4$.

The equivalence classes of Hadamard matrices of order ≤ 28 have been determined by Hall, Ito-Leon-Longyear and the author([5], [6], [7], [11] and [12]). There are exactly 487 inequivalent Hadamard matrices of order 28. One of them has no Hall set and the others have Hall sets. These matrices are distinguished by their K-matrices except five matrices in [9].

Let $F = GF(2)$ be the field of two elements 0 and 1. Let F^n be the vector space of dimension n over F. For elements $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ of F^n, the

Hamming distance $d(x,y)$ is defined by the number of i with $x_i \neq y_i$. The Hamming weight $wt(x)$ of x is defined by $d(x,0)$. For a column vector x put $wt(x) = wt(x^t)$. A binary linear (n,k) code C is a subspace of F^n of dimension k. The minimum distance of C is defined by the minimum weight among all non-zero elements of C. The generator matrix of C is the matrix whose rows are the basis vectors of C. Two codes are equivalent if one can be obtained from the other by a permutation of the coordinate positions.

We assume that the reader is familiar with the basic facts from the theory of self-dual linear codes. Our terminology follows [3].

It is well known that for a Hadamard matrix of order n there exists a binary code with n symbols, $2n$ code words, and minimum distance $n/2$. This is not necessarily a linear code. On the other hand many linear codes can be constructed from Hadamard matrices. It is well known that the $(24,12,8)$ Golay code is obtained from a Hadamard matrix of order 12 ([15]).

In [19] Tonchev gave a general method of a construction of binary self-orthogonal codes and in [2] obtained six doubly even self-dual $(56,28,12)$ codes from two Hadamard matrices of order 28. But two of the six codes were not distinguished.

We discuss the existence and equivalence of extremal doubly even self-dual $(56,28,12)$ codes obtained from all Hadamard matrices of order 28 by the method in [19].

We can consider that $(H+J)/2$ is a matrix on F, where J is the all $1's$ matrix, and we denote this also by H if there is no confusion.

Theorem 1. *Let H be a Hadamard matrix of order 28 and C a binary self-dual $(56,28,12)$ code with generator matrix (I,H). Then H and C are equivalent to one of two matrices and one of six codes in [2], respectively. Moreover two of the six codes are equivalent.*

One of the matrices in Theorem 1 is of $QR-type$ and the other is equivalent to the $471-th$ matrix in [13].

2. General properties

Let $H = (h_{i,j})$ be a normalized Hadamard matrix of order $n = 28$. Let $\Gamma = \{1,\cdots,28\}$. Let B and P be subsets of Γ. Let $H_{B,P}$ be a Hadamard matrix obtained from H by negating the rows in B and the columns in P. By [2], if the matrix $(I, H_{B,P})$ generates a binary self-dual doubly even $(56,28,12)$ code, then the following condition must be satisfied:

Condition 1. The weight of every row and column of $H_{B,P}$ is greater than 10 and congruent to 3 $(mod\ 4)$.

Let $C\{H\}$ be the set of equivalence classes of codes constructed from H as above. Then we have the following proposition.

Proposition 2. *If H' is equivalent to H, then $C\{H\} = C\{H'\}$.*

Therefore we may assume that H is of normal form, when we determine $C\{H\}$.

Proposition 3. *If B and P contain 1, then there exist subsets B' and P' of Γ not containing 1 such that $C(H_{B,P})$ is equivalent to $C(H_{B',P'})$.*

Proof. Put $B = 1 - B$ and $P = 1 - P$. Then the proposition follows. □

Since a matrix (H^t, I) generates $C(I, H)$, the following proposition is trivial.

Proposition 4. *$C\{H\} = C\{H^t\}$ and hence we assume $|B| \geq |P|$.*

We denote a code generated by $(I, H_{B,P})$ $EC(H_{B,P})$ if it is an extremal self-dual doubly even $(56, 28, 12)$ code.

We study H when $H_{B,P}$ generates $EC(H_{B,P})$. Since H is Hadamard matrix, it is trivial that the weight of a sum of every different two rows of $(I, H_{B,P})$ is 16. By Proposition 4, $(I, H_{B,P})$ generates $EC(H_{B,P})$, if Condition 1 and the following condition are satisfied:

Condition 2. Weights of sums of three or four rows of $(I, H_{B,P})$ and $(H^t_{B,P}, I)$ must be greater than 11, respectively.

At first we assume $B = \{2, \cdots, 28\}$ and $P = \{1\}$. Then $H_{B,P}$ satisfies the Condition 1 and hence $(I, H_{B,P})$ generates a doubly even code. Let H' be an equivalent matrix of H of normal form. Then a code generated by $(I, H'_{B,P})$ is equivalent to the above code by [16]. If H has no Hall set, then H^t has also no hall set by [14]. Thus the weights of sums of all three or four row vectors of H and H^t are greater than, or equal to 12. By Condition 2 $(I, H_{B,P})$ generates $EC(H_{B,P})$. In this case set $H_1 = H_{B,P}$. If H has Hall sets, then we may assume by Propositon 2 that a submatrix of H consisting of its first four rows is of form (1.1). Thus the sum of the first four rows of $(I, H_{B,P})$ is of weight 8 and the minimum weight of the code generated by $(I, H_{B,P})$ is less than 12.

Proposition 5. *If the weight of some column of $H_{B',P'}$ equals 27, then there exists a row of weight 27 and hence it is equivalent to $(I, H_{B,P})$.*

Proof. We may assume the weight of the first column equals 27, $h_{1,1} = 0$ and $h_{1,i} = 0$. By the orthogonality of the first and second columns, the weight of the second column must be 13. This contradicts the Condition 1. Hence $h_{1,i} = 1$ for all i. This proves the proposition. □

By this proposition and [16], if there exists a row or column of $H_{B,P}$ whose weight is 27, then the code generated by $(I, H_{B,P})$ is equivalent to the code generated by (I, H_1). Therefore we assume the weights of all rows and columns of $H_{B,P}$ are less than 27.

Since H does not have a Hall set, the weight of every different four rows of $(I, H_{B,P})$ and $(H^t_{B,P}, I)$ is greater than 12, respectively.

3. On the case H has no Hall set

In this section we assume that H has no Hall set. Then we may assume that H is the Paley matrix defined by the squares in $F = GF(27)$ by [12]. Let $F = Z_3[X]/(f(X))$, where $Z_3 = GF(3)$ and $f(X)$ is an irreducible polynomial over Z_3. We assign a number to an element of F in the following way:

$$aX^2 + bX + c \ (mod\ (f(x)) \longleftrightarrow 3^2a + 3b + c. \tag{3.1}$$

Then $D = \{1, 6, 7, 8, 9, 11, 12, 13, 15, 16, 20, 22, 25\}$ is a difference set and $\{i + D; i \in F\}$ is a set of blocks of a Hadamard $2-(27, 13, 6)$ design. We also denote an incidence matrix of this design by $\bar{D} = (d_{i,j})$. Then

$$H = (H_1 =) \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \bar{D} & \\ 1 & & \end{bmatrix}. \tag{3.2}$$

Let $col(D)$ be the set of columns i such that $(i-2)$ are not contained in the difference set D. The permutation groups $Aut(H)$ on the set of rows and $Aut(H)$ on the set of columns are same as permutation groups.

Proposition 6. *We may assume B and P do not contain 1.*

Proof. By Proposition 3 and 4, assume $1 \in B$ and $1 \notin P$. Since $1 < |B| \equiv 1\ (4)$ and $|P| \equiv 3\ (4)$, there exists $i\ (> 1)$ not contained in $B \cap P$. Then (i,i)-component of $H_{B,P}$ is 1. By Proposition 2 we may assume that $(1,1)$-component of $H_{B,P}$ is 1. This proves the proposition. □

Proposition 7. *We may assume B and P do not contain 2.*

Proof. If $B \cup P \neq \Gamma$, then the proposition follows. Assume $B \cup P \neq \Gamma$. Then, by the same permutation of Γ if necessary, $H_{B,P}$ is the following form:

$$H_{B,P} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & M_1 & & A_1 \\ 1 & & M_2 & B_1 \\ 0 & A_2 & B_2 & M_3 \end{bmatrix},$$

where M_1 and M_2 have diagonals of all $1's$, M_3 has a diagonal of all $0's$, $A_1^t = A_2$ and $B_1^t = B_2$. Let M_i be of size m_i $(i = 1, 2, 3)$. If $m_3 = 0$, then $m_1 + 1 = 11, 15, \cdots$ and $m_2 + 1 = 11, 15, \cdots$. this contradicts $m_1 + m_2 = 27$. Therefore $m_3 > 0$. If $A_1 = 0$ and $B_1 = 0$, then $m_3 \geq 12$ by Condition 2. Thus there exists a pair (i,j) of a row and a column such that the $(i,i), (i,j)$ and (j,i) components are 1 and the (j,j) component is 0. Since $Aut(H)$ is doubly transitive, then the proposition follows. □

Proposition 8. *$|col(D) - P|$ is an odd number greater than 2 and we may assume P does not contain 3.*

Proof. Let a and b be the numbers of columns of $col(D) - P$ and columns not contained in $P \cup col(D) \cup \{1, 2\}$, respectively. Applying Condition 1 to the first and the second columns, $a + b = 9, 13, \cdots$ and $a + (13 - b) = 10, 14, \cdots$. $a + b \equiv 1\ (4)$ and $a - b \equiv 1\ (4)$. Therefore $a \geq 3$ and a is odd. There exists an element of $Aut(H)$ fixing the first and second rows and columns such that it transforms the $3rd$ column to a column of $col(D) - P$. Thus we may assume the $3rd$ column is not negated. □

Under Conditions 1 and 2 we compute by a computer program satisfying the propositions in this section. Then we have four solutions case 2, 3, 4 and 5 in Table 2.

In this section we assume that H has Hall sets. Then H is one of the 486 matrices obtained in [11] by Proposition 2. The following proposition is trivial from the definition of Hall set.

Proposition 9. *Let $\{r_1, r_2, r_3, r_4\}$ be a Hall set of H. The vector $r_1 + r_2 + r_3 + r_4$ in F^n is of weight 4 or 24.*

By this, if $(I, H_{B,P})$ generates $EC(H_{B,P})$, then

$$|B \cap \{r_1, r_2, r_3, r_4\}| = \begin{cases} 1 \text{ or } 3, \text{ if } wt(r_1 + r_2 + r_3 + r_4) = 4 \\ 0, 2 \text{ or } 4, \text{ if } wt(r_1 + r_2 + r_3 + r_4) = 24. \end{cases} \tag{4.1}$$

Let $\{\Lambda_1, \cdots, \Lambda_m\}$ be a family consisting of all the Hall sets of H. Set $\Sigma_i = \Lambda_1 \cup \cdots \cup \Lambda_i$. Assume the following:

Condition 3. $\Sigma_1 \subsetneq \Sigma_2 \subsetneq \cdots \subsetneq \Sigma_k = \Sigma_m$

Set $k(H) = \Gamma - \Sigma_m$ for H. For almost all matrices in [11] $\Sigma = \Gamma$. $Max\{k(H)\} = 3$. If we make a computer program under the conditions in Section 2 and this section, H must be a matrix H_{471} in [11], say H_2 in this paper. In fact the following subfamily of Hall sets of H_2 satisfies Condition 3:

TABLE 1. A subfamily of Hall sets of H_2

Hall set	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8	Λ_9	Λ_{10}
rows	2	2	2	3	3	3	4	4	4	5
	11	12	15	12	13	16	11	13	14	14
	21	23	21	22	24	22	23	20	20	23
	24	24	23	25	25	24	25	23	25	27
weight	4	4	4	4	4	4	4	4	4	4

Hall set	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	Λ_{17}	Λ_{18}	Λ_{19}	
rows	5	5	6	6	7	7	8	9	10	
	15	17	15	18	14	19	12	13	11	
	26	23	24	24	26	25	21	22	20	
	27	26	28	27	28	28	26	27	28	
weight	4	4	4	4	4	4	4	4	4	

The weight of the sum of the four elements of every Hall set Λ_i $\{i = 1, \cdots, 19\}$ is four. Therefore $|\{B \cap \Lambda_i\}|$ must be one or three. There exists no Hall set containing the first row and the first column is not contained in any Hall set of H^t. In this case we may compute by hand and two solutions are obtained:

$$\begin{cases} B_1 = \{2, 3, 4, 5, 6, 7, 8, 9, 10\} \\ P_1 = \{11, 12, 13, 14, 15, 16, 17, 18, 19\} \end{cases} \tag{4.2}$$

and

$$\begin{cases} B_2 = \{11, 12, 13, 14, 15, 16, 17, 18, 19\} \\ P_2 = \{2, 3, 4, 5, 6, 7, 8, 9, 10\} \end{cases} \tag{4.3}$$

Since H_2 and H_2^t have the same $K-$matrix, they are equivalent. In fact, by the following permutations g and h on the sets of rows and columns of H_2, respectively, H_2 transforms to its transpose.

(4.4)
$$g = (2,3)(4,9)(5,8)(6,7)(10,11)(13,17,15,18,14,16)(19,20)(22,25,23,27,24,26)$$
(4.5)
$$h = (2,3)(4,9)(5,8)(6,7)(10,11)(13,16,14,18,15,17)(19,20)(22,26,24,27,23,25)$$

Applying g and h, (4.2) is equivalent to (4.3). This code is the 6-th code in Table 2.

5. On equivalence of codes

In this section we discuss the equivalence of the six codes in Table 2 or in [2]. We consider the 3−designs formed by the minimum weight codewords. The definition of the class size of a code is given in [2]. The codes $No.4$ and $No.6$ in Table 2 have the same class size.

We introduced the $K-$matrices and K-boxes associated with Hadamard matrices. This idea is useful for a classification of codes or designs. For the definition of $K-$matrix, see [8] or [9]. Let C be a binary self-dual doubly even $(56, 28, 12)$ code. We define the $K-$box for C. Let $D = (d_{i,j})$ be the $3-(56, 12, 65)$ design formed by the minimum weight codewords.

For any different p points $i_1, \cdots, i_p$, let $b_{i_1,\cdots,i_p}$ be the number of blocks of D containing the p points. Next we define $a_{i_1,\cdots,i_p}$ for two positive integers q_1 and q_2 $(q_1 \leq q_2)$ as follows:

$$a_{i_1,\cdots,i_p}(q_1,q_2) = \begin{cases} 1, & \text{if } q_1 \leq b_{i_1,\cdots,i_p} \leq q_2 \\ 0, & \text{otherwise} \end{cases}$$

If $|\{i_1, \cdots, i_p\}| < p$, then set $a_{i_1,\cdots,i_p} = 0$. For three points x, y and z, define $c_{x,y,z}(q_1, q_2)$ as follows:

$$c_{x,y,z}(q_1,q_2) = \sum_{x,y,z \in \{i_1,\cdots,i_p\}} a_{i_1,\cdots,i_p}(q_1,q_2).$$

For fixed i, j, by a permutation of indexes we assume that $c_{k_1,i,j}(q_1,q_2) \leq c_{k_2,i,j}(q_1,q_2)$ if $k_1 < k_2$. Then we have 56 matrices $B(i)''_{p,q_1,q_2}(C) = (c_{j,k,i}(q_1,q_2)), i = 1, \cdots, 56$. For fixed i, after ordering $B(i)''_{p,q_1,q_2}(C)$ lexicographically, we denote this by $B(i)'_{p,q_1,q_2}(C)$. Furthermore we rearrange lexicographically the collection of matrices $B(i)'_{p,q_1,q_2}$ with $1 \leq i \leq 56$. We call this collection the $K-$box of C (or D) and denote it by $B_{p,q_1,q_2}(C)$. By the definition of $K-$box, if a code C' is equivalent to C, then their $K-$boxes are the same.

Let C_4 and C_6 be codes $No.4$ and $No.6$ in Table 2, respectively. Then $B_{5,1,8}(C_4) = B_{5,1,8}(C_6)$ and therefore we can not distinguish the two codes by $K-$boxes. But we can find a permutation g of coordinates of C_4 such that C_4 is equivalent to

C_6 by g. At first we can find a permutation g_1 of the set $\{1,\cdots,56\}$ such that $B(i)_{5,1,8}(C_4^{g_1}) = B(i)_{5,1,8}(C_6)$ for all i:

$$g_1 = \begin{cases} (1)(2,20,26,54,17,33,32,31,9,19,25,51,53,16,30,39,42, \\ 38,41,37,34,40,3,35,14,6,21,48,45,8,56,47,44,7,36,15, \\ 24,27,4,18)(5,10,22,11,23,13)(12,50,52,46,49)(28)(29)(55) \end{cases}$$

Next we find a permutation g_2 such that $B(2)'_{5,1,8}(C_4^{g_1g_2}) = B(2)'_{5,1,8}(C_6)$:

$$g_2 = \begin{cases} (1)(2)(3,12,17,14,19,10,7,18,8,15,16,13,6,5,4)(9,11) \\ (20,42,25,41,22,45)(21,39,26,47,27,40,28,44,23,43)(24,46) \\ (29)(30,56,52,37,38,36,48,51,53,33,55,34,32,35,49,31)(50,54). \end{cases}$$

$G = (I, H_4)^{g_1g_2}$ is a generator matrix of the code $C_4^{g_1g_2}$. We can obtain another generator matrix (I, H) from G. Then we can easily check that H is an Hadamard matrix with Hall sets and its $K-$matrix is one of H_2 in (5.1). Therefore H is equivalent to H_2 by [11]. Thus C_4 is equivalent to C_6.This completes the proof of Theorem 1.

TABLE 2. The extremal codes

code	H	Negated columns (P) and rows (B)	Class sizes
1	H_1	col: 1 row: 2,$\cdots$,28	56
2	H_1	col: 4, 5, 6, 7, 8,12,17,19,28 row: 3, 6, 8, 9,10,11,13,19,20,22,25,26,28	2,6,6,6,6,6,6,6,6,6
3	H_1	col: 4,5,8,21,22 row: 12,14,17,21,22	2,2,2,2,6,6,6,6,6,6,6
4	H_1	col: 4, 5, 9,13,14,18,22,23,27 row: 3, 7, 8,12,16,17,21,25,26	2,18,18,18
5	H_1	col: 4, 5, 9,11,13,14,15,18,19 row: 12,16,17,21,22,23,25,26,27	1,1, 9,9,9,9,9
6	H_2	col: 3,4,5,6,7,8,9,10,11 row: 12,13,14,15,16,17,18,19,20	2,18,18,18

(5.1)

$$H_1 = \begin{bmatrix}
1\ 111\ 111\ 111\ 111\ 111\ 111\ 111\ 111\ 111 \\
1\ 010\ 000\ 111\ 101\ 110\ 110\ 001\ 010\ 010 \\
1\ 001\ 000\ 111\ 110\ 011\ 011\ 100\ 001\ 001 \\
1\ 100\ 000\ 111\ 011\ 101\ 101\ 010\ 100\ 100 \\
1\ 111\ 010\ 000\ 110\ 101\ 110\ 010\ 001\ 010 \\
1\ 111\ 001\ 000\ 011\ 110\ 011\ 001\ 100\ 001 \\
1\ 111\ 100\ 000\ 101\ 011\ 101\ 100\ 010\ 100 \\
1\ 000\ 111\ 010\ 110\ 110\ 101\ 010\ 010\ 001 \\
1\ 000\ 111\ 001\ 011\ 011\ 110\ 001\ 001\ 100 \\
1\ 000\ 111\ 100\ 101\ 101\ 011\ 100\ 100\ 010 \\
1\ 001\ 010\ 010\ 010\ 000\ 111\ 101\ 110\ 110 \\
1\ 100\ 001\ 001\ 001\ 000\ 111\ 110\ 011\ 011 \\
1\ 010\ 100\ 100\ 100\ 000\ 111\ 011\ 101\ 101 \\
1\ 010\ 001\ 010\ 111\ 010\ 000\ 110\ 101\ 110 \\
1\ 001\ 100\ 001\ 111\ 001\ 000\ 011\ 110\ 011 \\
1\ 100\ 010\ 100\ 111\ 100\ 000\ 101\ 011\ 101 \\
1\ 010\ 010\ 001\ 000\ 111\ 010\ 110\ 110\ 101 \\
1\ 001\ 001\ 100\ 000\ 111\ 001\ 011\ 011\ 110 \\
1\ 100\ 100\ 010\ 000\ 111\ 100\ 101\ 101\ 011 \\
1\ 101\ 110\ 110\ 001\ 010\ 010\ 010\ 000\ 111 \\
1\ 110\ 011\ 011\ 100\ 001\ 001\ 001\ 000\ 111 \\
1\ 011\ 101\ 101\ 010\ 100\ 100\ 100\ 000\ 111 \\
1\ 110\ 101\ 110\ 010\ 001\ 010\ 111\ 010\ 000 \\
1\ 011\ 110\ 011\ 001\ 100\ 001\ 111\ 001\ 000 \\
1\ 101\ 011\ 101\ 100\ 010\ 100\ 111\ 100\ 000 \\
1\ 110\ 110\ 101\ 010\ 010\ 001\ 000\ 111\ 010 \\
1\ 011\ 011\ 110\ 001\ 001\ 100\ 000\ 111\ 001 \\
1\ 101\ 101\ 011\ 100\ 100\ 010\ 000\ 111\ 100
\end{bmatrix}$$

$$H_2 = \begin{bmatrix}
1\ 111\ 111\ 111\ 111\ 111\ 111\ 111\ 111\ 111 \\
1\ 100\ 010\ 110\ 000\ 001\ 101\ 111\ 010\ 110 \\
1\ 010\ 001\ 011\ 000\ 100\ 110\ 111\ 001\ 011 \\
1\ 001\ 100\ 101\ 000\ 010\ 011\ 111\ 100\ 101 \\
1\ 011\ 001\ 010\ 101\ 000\ 001\ 110\ 111\ 100 \\
1\ 101\ 100\ 001\ 110\ 000\ 100\ 011\ 111\ 010 \\
1\ 110\ 010\ 100\ 011\ 000\ 010\ 101\ 111\ 001 \\
1\ 001\ 110\ 001\ 001\ 101\ 000\ 100\ 011\ 111 \\
1\ 100\ 011\ 100\ 100\ 110\ 000\ 010\ 101\ 111 \\
1\ 010\ 101\ 010\ 010\ 011\ 000\ 001\ 110\ 111 \\
1\ 000\ 100\ 110\ 111\ 100\ 110\ 100\ 100\ 110 \\
1\ 000\ 010\ 011\ 111\ 010\ 011\ 010\ 010\ 011 \\
1\ 000\ 001\ 101\ 111\ 001\ 101\ 001\ 001\ 101 \\
1\ 011\ 000\ 100\ 110\ 111\ 100\ 110\ 010\ 001 \\
1\ 101\ 000\ 010\ 011\ 111\ 010\ 011\ 001\ 100 \\
1\ 110\ 000\ 001\ 101\ 111\ 001\ 101\ 100\ 010 \\
1\ 001\ 011\ 000\ 010\ 011\ 111\ 100\ 101\ 010 \\
1\ 100\ 101\ 000\ 001\ 101\ 111\ 010\ 110\ 001 \\
1\ 010\ 110\ 000\ 100\ 110\ 111\ 001\ 011\ 100 \\
1\ 111\ 010\ 011\ 100\ 001\ 110\ 000\ 100\ 101 \\
1\ 111\ 001\ 101\ 010\ 100\ 011\ 000\ 010\ 110 \\
1\ 111\ 100\ 110\ 001\ 010\ 101\ 000\ 001\ 011 \\
1\ 110\ 111\ 001\ 011\ 010\ 100\ 110\ 000\ 100 \\
1\ 011\ 111\ 100\ 101\ 001\ 010\ 011\ 000\ 010 \\
1\ 101\ 111\ 010\ 110\ 100\ 001\ 101\ 000\ 001 \\
1\ 100\ 101\ 111\ 100\ 011\ 010\ 100\ 011\ 000 \\
1\ 010\ 110\ 111\ 010\ 101\ 001\ 010\ 101\ 000 \\
1\ 001\ 011\ 111\ 001\ 110\ 100\ 001\ 110\ 000
\end{bmatrix}$$

$$K(H_2) = \begin{bmatrix}
1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
9 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 111 & 222 \\
9 & \dots & \dots & \dots & \dots & \dots & \dots & 111 & 111 & 111 \\
9 & \dots & \dots & \dots & \dots & 111 & 111 & 111 & 111 & 222
\end{bmatrix}$$

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