FRACTALS, RECURSIONS, DIVISIBILITY

by

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Abstract: The self similar (fractal) structure of the Pascal triangle of binomial coefficients modulo some fixed prime has been known for some time and explored in detail. It turns out that such a fractal structure is the common property of arithmetical functions of two variables possessing a more general recursive relation

$$(R) \qquad f(i,j) = af(i-1,j) + bf(i-1,j-1)$$

where a and b are fixed integers.

When a, b are functions of i and j, as for example in the case of Gaussian binomials, or Stirling numbers of first or second kind, or when the recursion is of type

$$f(i,j) = af(i-1,j) + bf(i-1,j-1) + cf(i-1,j+1) + df(i,j-1)$$

more complexity is shown in the modulo p arrays.

Functions defined by the recursion R, two dimensional lattice walks and Stirling numbers produce arrays modulo p which will be described in this paper.

1. INTRODUCTION

Fractals as introduced by B. Mandelbrot (1975) [4] are sets characterised by a non-integer Hausdorff dimension. Such sets abound in nature. Cloud formations, shell designs, snow flakes, rugged shore lines have fractal structure. Fractal sets are also constructed mathematically. Such a set is for example the Koch curve, simulating the promontories and inlets of an idealised shore line. Figure 1 gives the first few steps of its construction.

Figure 1

Another well known fractal set is the Cantor set, a variant of which of particular interest here is the Sierpinski gasket. Its construction begins (figure 2) by bisecting each side of an equilateral triangle obtaining thus four congruent triangles. The middle triangle is then removed and the construction is repeated on the remaining triangles. Figure 2 shows the result of three iterations, the removed triangles being darkened.

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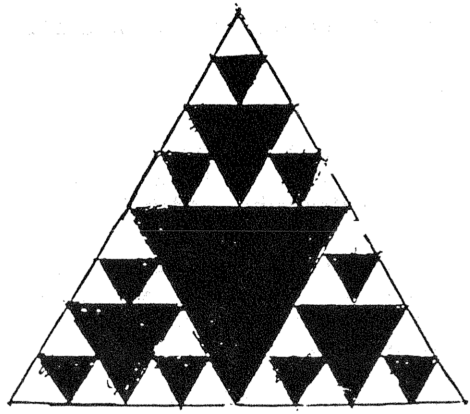

Figure 2

The Sierpinski gasket is the set of points remaining after an infinite number of repetitions of the procedure.

The Hausdorff dimensions of the Koch curve and the Sierpinski gasket can be calculated. They are found to be log 4/log 3 and log 3/log 2 respectively. See e.g. [1].

Table 1 shows the array of binomial coefficients modulo 2. Here as well as in other tables to follow, zero entries are marked by dots to make them more conspicuous. Comparison with figure 2 shows the intimate connection of this array with the Sierpinski gasket. Indeed, k iterations in the approximating constructions of the latter, produce a structure identical to the array

$$\left(\binom{n}{r} \bmod 2\right), \quad 0 \leq n < 2^{k+1}.$$

In [11], S. Wolfram examined the fractal structure of this binomial array and found that its Hausdorff dimension is log 3/log 2, the same as that of the Sierpinski gasket.

While the exact definition and characterisation of fractal sets is given in terms of dimension and measure, it is a feature shared by a large number of fractal sets which makes them mathematically so interesting and aesthetically attractive. This property is *self similarity*. It means that the sets are built up of pieces geometrically similar to the entire set, but on a smaller scale. Figures 1 and 2 show clearly this property, and so does the binomial array modulo 2, having the same structure as the Sierpinski gasket.

Table 2 shows the binomial array modulo 5. It displays the structure of the binomial array modulo a prime number p, more clearly than the special modulo 2 array. These arrays are analysed in detail in [6] and [7], with all the necessary proofs. Here only some definitions and names of the most important partial arrays are given for further use.

(a) *Zero holes.* These are inverted triangles, containing exclusively zero entries (marked by dots). They are given orders $1, 2, \ldots$ according to their sizes. It follows from the Pascal recursion formula, (applied modulo p)

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \pmod p \tag{1}$$

that a string of zero entries in a row generates a string shortened by one entry in the next row.

(b) *Principal cell.* This is the array of the entries

$$\left(\binom{n}{r} \bmod p\right) \quad \text{for} \quad 0 \leq n < p.$$

The *head* of this triangular array is the entry

$$\binom{0}{0} = 1 \quad \text{(modulo any prime)}.$$

It follows again from the recursive formula (1) that between two neighbouring zero-holes a triangular array arises which is *similar* to the principal cell. If the entry at the head of this triangle is h, then in the next row we have $h \quad h$, followed by $h \quad 2h \quad h$, and

$$h \quad \binom{n}{1}h \quad \ldots \quad \binom{n}{n-1}h \quad h \quad \pmod p$$

in the $(n+1)^{\text{th}}$ row, where $0 \leq n < p$. Thus we have a cell.

(c) *Cell.* A triangular structure with entries proportional to those of the principal cell.

(d) *Principal Cluster.* The array of entries

$$\left(\binom{n}{r} \bmod p\right) \quad \text{for} \quad 0 \leq n < p^m,$$

where m is called the order of the principal cluster.

(e) *Cluster.* A triangular structure *similar* to the principal cluster, the entries being h times the corresponding entries of the principal cluster of the same order, where h is the entry at the head of the cluster.

Table 2 shows clearly the layered structure of each cluster. Each m-cluster consists of p layers of $(m-1)$ clusters alternating with corresponding zero-holes.

Notation: For brevity, here and in the following sections the notation

$$\overline{a} = a \pmod p,$$

will be used whenever the prime number p is fixed. Thus

$$\overline{\binom{n}{r}} = \binom{n}{r} \pmod p.$$

The theorem of Lucas (1878), made better known through the expository article of Fine [2], enables us to calculate general entries $\overline{\binom{n}{r}}$.

Let

$$\left.\begin{aligned} n &= a_m p^m + a_{m-1}p^{m-1} + \ldots + a_0 \\ r &= b_m p^m + b_{m-1}p^{m-1} + \ldots + b_0 \end{aligned}\right\} \tag{2}$$

where $0 \le a_i, b_i < p$ $(0 \le i \le m)$.

Then

$$\overline{\binom{n}{r}} = \overline{\binom{a_m}{b_m}} \cdot \overline{\binom{a_{m-1}}{b_{m-1}}} \cdots \overline{\binom{a_0}{b_0}}. \tag{3}$$

Though Lucas obtained this result algebraically, the theorem has a simple geometrical interpretation.

Each entry of the binomial mod p array is positioned inside a nest of clusters of order $m, m-1, \ldots, 1$ respectively. Beginning with the largest cluster and moving inwards, the heads of these clusters are

$$\overline{\binom{a_m}{b_m}}, \overline{\binom{a_m}{b_m}}, \overline{\binom{a_{m-1}}{b_{m-1}}}, \ldots, \overline{\binom{a_1}{b_1}} \text{ in order.}$$

From these, (using the similarity principle), (3) follows. (For more detail, see [6] or [7].)

Thus the entries of the principal cell of the mod p array determine the whole of the array.

Note: P. Goetgheluck [3] considers *all* prime divisors of $\binom{n}{r}$ for a *fixed n*. An attractive three dimensional plot of the variables (p, n, r) produces sets of zero holes as two dimensional projections.

The question then arises naturally. Is this fractal structure the exclusive property of the binomial coefficients?

The basic ingredient of all the proofs establishing the structure of the binomial p array is the recurrence relation (1). More general recurrence relations define other arithmetical functions. Arrays modulo p associated with such functions were investigated. Some examples will be described in the following sections, with many others still left open.

2. GENERALISATION OF THE BINOMIAL RECURSION FORMULA

Table 3 shows the array $(f(i,j) \pmod 5)$ where the function $f(i,j)$ is defined by the recursion

$$f(i,j) = 3f(i-1, j-1) + 2f(i-1, j).$$

Apart from the values of the non-zero entries the array does not differ from the $\binom{n}{r}$ mod 5 array.

In general consider the function

$$\left.\begin{aligned} &f(i,j) = af(i-1,j-1) + bf(i-1,j) \\ &f(0,0) = 1 \\ &f(0,x) = 0 \quad \text{when} \quad x \neq 0. \end{aligned}\right\} \tag{4}$$

where a, b are integers, not divisible by p.

Introduce the shift operators

$$\begin{aligned} \sigma_1 f(i,j) &= f(i-1, j-1) \\ \sigma_2 f(i,j) &= f(i-1, j), \end{aligned}$$

writing (4) in the form

$$f(i,j) = (a\sigma_1 + b\sigma_2) f(i,j). \tag{5}$$

It is easy to see that σ_1 and σ_2 commute, iterations (products) are associative, so that $\sigma_1^s\sigma_2^t$ is well defined.

$$\sigma_1^s\sigma_2^t f(i,j) = f(i-(s+t), j-s). \tag{6}$$

Thus iterating (5), and using (6) we obtain

$$(a\sigma_1 + b\sigma_2)^n f(i,j) = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r f(i-n, j-(n-r)).$$

Setting $n = i$, and the initial values as in (4), obtain

$$f(i,j) = (a\sigma_1 + b\sigma_2)^i f(i,j) = \binom{i}{i-j} a^j b^{i-j},$$

whence

$$f(i,j) = \binom{i}{j} a^j b^{i-j}. \tag{7}$$

It follows immediately from (7) that the zero-hole structure of the array $f(i,j)$ (mod p) is identical with that of the $\binom{i}{j}$ mod p array, while the heads of the cells and clusters of the former take values $\overline{\binom{i}{j} a^i b^{i-j}}$ replacing $\overline{\binom{i}{j}}$.

The general entry

$$\overline{f(n,r)} = \overline{a^r b^{n-r} \binom{n}{r}}$$

can be found directly by the Lucas formula for $\overline{\binom{n}{r}}$, multiplied by

$$\overline{a^r b^{n-r}} = a^R b^{N-R}$$

where

$$R \equiv r \pmod{(p-1)} \quad \text{and} \quad N \equiv n \pmod{(p-1)}$$

by Fermat's theorem.

3. LATTICE WALKS

A lattice walk is a path from the origin (0,0) to the point (i,j) where i,j are positive integers, consisting of steps of unit length in directions parallel to the axes. Here we restrict ourselves to steps taken in the positive direction. The solution of the problem of finding the number $P(i,j)$ of all possible paths is well known. Here we point out that the expression for $P(i,j)$ can also be found by using the recursion

$$\left.\begin{aligned} P(0,0) &= 1. \\ P(i,j) &= P(i-1,j) + P(i,j-1) \\ &= (\sigma_1+\sigma_2)P(i,j) \end{aligned}\right\} \tag{8}$$

where the shift operators this time are

$$\sigma_1 P(i,j) = P(i-1,j)$$
$$\sigma_2 P(i,j) = P(i,j-1)$$

and

$$\sigma_1^s\sigma_2^t P(i,j) = P(i-s, j-t),$$

hence

$$P(i,j) = (\sigma_1+\sigma_2)^n P(i,j) = \sum_{r=0}^{n} \binom{n}{r} P(i-r, j-(n-r)). \tag{9}$$

Since $n = i+j$ steps are needed to get from (0,0) to (i,j), we have

$$P(i,j) = \sum_{r=0}^{i+j} \binom{i+j}{r} P(i-r, r-i),$$

where $P(i-r, r-i) = 0$ unless $r = i$, hence

$$P(i,j) = \binom{i+j}{i} = \binom{i+j}{j}.$$

Table 4 shows the array

$$(P(i,j) \mod 5).$$

Not surprisingly, the only difference between its geometry and that of the binomial array, is in the direction of the layers of clusters and zero-holes, rotated by an angle of 45°, the result of the transformation

$$(i,j) \to (i+j, j).$$

Weighted paths

Table 5 gives an example for an array of a *weighted* lattice walk, modulo a prime. It shows the array

$$(P'(i,j) \mod 5),$$

where

$$P'(i,j) = 2P'(i-1,j) + 3P'(i,j-1).$$

More generally, a *weighted lattice walk* is defined by the recursion

$$P'(i,j) = aP'(i-1,j) + bP'(i,j-1)$$
$$P'(0,0) = 1$$

where a and b are constant integers.

For calculating $P'(i,j)$ equation (9) is modified to

$$P'(i,j) = (a\sigma_1 + b\sigma_2)^n P'(i,j) = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r P'(i-r, j-(n-r))$$

giving the result

$$P'(i,j) = \binom{i+j}{i} a^j b^i$$

in accordance with result (7) in Section 2. Thus the $(P'(i,j) \mod p)$ array is obtained from the $f(i,j)$ array of Section 2 by a 45 degree rotation. The calculation of the general $\overline{P(i,j)}$ or $\overline{P'(i,j)}$ entries is similar to the calculation used in the case of the binomial, or more general linear recursion, replacing in (7) i by $i+j$.

Restricted walks

The function $R(i,j)$ is defined similarly to $P(i,j)$, by recursion formula (8), but subject to the restriction that $j \leq i$, that is

$$R(i,j) = 0 \quad \text{when} \quad j > i.$$

Table 6 shows the array

$$(R(i,j) \bmod 5).$$

While strict self-similarity of cells and clusters no longer holds in these arrays, there are still some interesting features: zero-holes of the $(P(i,j) \bmod p)$ array seem to be preserved, and in addition, regularly spaced diagonal strips of zero-entries appear. It is of some interest to analyse these arrays.

To determine the function $R(i,j)$, the known technique of *reflection* is used.

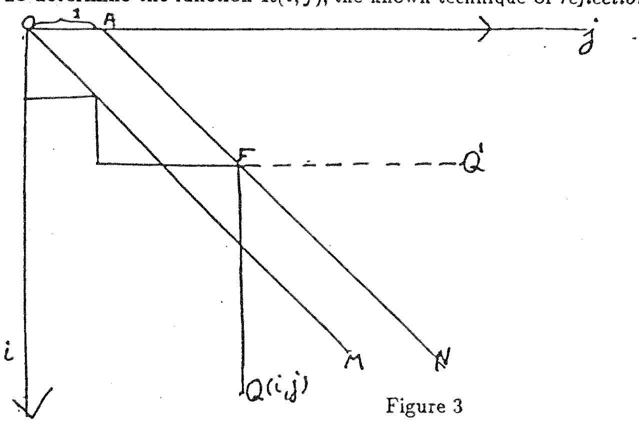

Figure 3

In Figure 3 the lines OM and AN are parallel. OM representing the boundary line $i = j$, while $j = i+1$ holds for points on AN. A path becomes "illegal" if it reaches the line AN, as the path shown in the figure. All such paths reaching the point $Q(i,j)$ must be excluded. Denote by F the point where the path first meets the line AN. Reflect the remaining path from F to Q about the line AN, (shown as a dotted section). It is easy to see that the reflection of Q is the point $Q'(j-1, i+1)$. Conversely, all paths ending at Q' can be reflected, obtaining an illegal path ending at Q.

Thus

$$R(i,j) = P(i,j) - P(j-1, i+1)$$

or using the obvious symmetry relation $P(a,b) = P(b,a)$, we obtain

$$R(i,j) = P(i,j) - P(i+1, j-1),$$

and by (9), applying the formula mod p, we obtain

$$\overline{R(i,j)} = \overline{\binom{i+j}{j}} - \overline{\binom{i+j}{j-1}} \tag{10}$$

Thus by Lucas' formula the values of $\overline{R(i,j)}$ can be evaluated for all i, j, using expansions of $i+j$, j and $j-1$.

Zero-holes of the $\overline{P(i,j)}$ array are initiated where

$$\overline{\binom{i+j}{j}} = 0, \quad \text{i.e. when} \quad i+j \equiv 0 \pmod{p}$$

and $j \not\equiv 0 \pmod p$.

For the $(\overline{R(i,j)})$ array we have *also*

$$\overline{\binom{i+j}{j-1}} = 0 \quad \text{when} \quad (i+j) \equiv 0 \bmod p)$$

and $j \not\equiv 0$ or $1 \pmod p$.

Hence the zero-holes of the $(\overline{P(i,j)})$ array are slightly reduced in the $(\overline{R(i,j)})$ array, being restricted by the additional condition $j \not\equiv 1 \pmod p$ (or $i \not\equiv -1 \pmod p$).

To account for the diagonal strips of zero-entries, write

$$\binom{i+j}{j} = \frac{i+1}{j}\binom{i+j}{j-1},$$

hence

$$j\binom{i+j}{j} \equiv (i+1)\binom{i+j}{j-1} \pmod p$$

It follows then from (10) that when

$$j \equiv (i+1) \bmod p \quad \text{and} \quad j \not\equiv 0 \bmod p, \tag{11}$$

then

$$\overline{R(i,j)} = 0.$$

Congruence (11) represents the equations of the diagonal strips.

4. STIRLING ARRAYS

In Sections 2 and 3 the recursion formulae (4) and (8) have constant coefficients, and so the shift operator method lends itself readily for finding an explicit expression for the arithmetic function. However, many arithmetic functions can be defined by recursions of different types, involving non constant coefficients. Such recursions may not be helpful for the purpose of finding general explicit formulae for these functions, but still provide the means of analysing arrays modulo p and finding Lucas type formulae for general entries modulo p. The best known examples for such functions are the Stirling numbers, kind 1 and kind 2.

Stirling numbers, kind 1

Define the function $S(n,k)$ by the recursion

$$\left.\begin{array}{l} S(n,k) = S(n-1,k-1) - nS(n-1,k) \\ S(0,i) = \delta_0^i \text{ that is } \left(S(0,i) = \begin{array}{ll} 0 & i \neq 0 \\ 1 & i = 0 \end{array}\right) \end{array}\right\} \tag{12}$$

Denote the generating polynomial P_n, where

$$P_n = \sum_{k=0}^{n} S(n,k)x^k. \tag{13}$$

Then using (12)

$$\begin{aligned} P_n &= \sum_{k=0}^{n}(S(n-1,k-1) - nS(n-1,k))x^k \\ &= xP_{n-1} - nP_{n-1} = (x-n)P_{n-1} \end{aligned}$$

and since by (12) and (13), $P_0 = 1$ we have

$$P_n = \prod_{j=1}^{n}(x-j).$$

Thus the coefficients of P_n are:

$$\left.\begin{array}{l} S(n,k) = (-1)^{n-k} \displaystyle\sum_{1 \le i_1 < i_2, \ldots < i_{n-k} \le n} i_1 i_2 \ldots i_{n-k} \text{ when } k < n \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{and } S(n,n) = 1 \end{array}\right\} \qquad (14)$$

Hence the Stirling numbers (kind 1) are determined by writing down the elementary symmetric functions with alternating signs.

A table (using 12) may be written as

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& -1 && 1 &&&& \\ &&& 2 && -3 && 1 &&& \\ && -6 && 11 && -6 && 1 && \\ & 24 && -50 && 55 && -10 && 1 & \\ \ldots && \ldots && \ldots && \ldots && \ldots && \ldots \end{array}$$

Table 7 shows the array of these Stirling numbers modulo 5: a slightly changed fractal structure, the first zero-hole beginning in the fifth row as indicated already in the table above.

Some features of this array are similar to the binomial array, but one conspicuous difference is the appearance of a progressively widening sheaf of zero-entries on the left side of the array after the 5th (generally after the p-th) row of the array.

Recursion formula (12) accounts easily for the zero-border of the array. We have from it that

$$\overline{S(ap,k)} = \overline{S(ap-1,k-1)} - ap\overline{S(ap-1,k)}. \qquad (15)$$

Thus for $a = 1$, $k = 0$, $S(ap-1,-1) = 0$ and so $p|S(p,0)$. (This is also clear from the fact that $S(n,0) = (-1)^n n!$, so for $n < p$, $p \not| S(n,0)$, but $p|S(p,0)$.)

It follows also immediately from (12) and (15) that the zero-border of the array widens by 1 if and only if $n = ap$.

Consider next the coefficients $S(p-1,k)$ (mod p). By (14) these are the elementary symmetric functions with alternating signs of the non-zero residues modulo p, hence equal to the coefficients of the equation

$$x^{p-1} - 1 = 0$$

in the field $GF(p)$. It follows that

$$\overline{S(p-1,k)} = 0 \quad \text{for} \quad 0 < k < p-1$$

while

$$\overline{S(p-1,0)} = -1 \quad \text{and} \quad \overline{S(p-1,p-1)} = 1. \qquad (16)$$

Thus the p-th row of the array ($n = p-1$) consists of zero-entries, except for its two extremities.

The *principal cell* of this Stirling array (mod p) is the array

$$(\overline{S(n,k)} \;|n < p-1).$$

However, relation (15) implies, that when $n = ap$, *the row of entries is an exact copy of row* $n = ap - 1$ *translated by one entry* (see Table 7). Thus new zero-holes begin, together with heads of new cells in rows where $n = ap$. As in binomial arrays and arrays discussed in Section 2, *cells, similar to the principal cell* border the zero-holes.

Next we look at the *heads* of the cells. Suppose that the heads of two neighbouring cells are h_1 and h_2. Then by similarity and relation (16) we have that the extremities of the bases of the cells in a row where $n = ap - 2$ for some a,

$$\text{are:} \quad -h_1 \;\; h_1 \quad \text{and} \quad -h_2, \;\; h_2 \;\; \text{respectively.}$$

If e is the entry in row $n = ap-1$, directly under the right extremity of the first and left extremity of the second cell, then by (12)

$$e = h_1 - (ap-1). - h_2 \equiv h_1 - h_2 \pmod p.$$

Since by the previous observation the head h of the corresponding cell in row $m = ap$, is equal to e, we have

$$h = h_1 - h_2. \tag{17}$$

It follows now by induction that the *heads* appearing in row $n = ap$ where $a < p$ *are binomial coefficients modulo p with alternating signs.*

More precisely, the set of head-entries for the row $n = ap$, $a < p$ is

$$\left\{(-1)^{a+i}\binom{a}{i} \bmod p \mid 0 \le i \le a\right\}. \tag{18}$$

It follows then that for $a = p - 1$, all the heads are equal to 1, so by (17), all entries in the row $n = p^2 - 1$ are equal to zero, with the exceptions

$$\overline{S(p^2-1, p-1)} = -1 \quad \text{and} \quad \overline{S(p^2-1, p^2-1)} = 1.$$

The array $(\overline{S(n,r)} \;|n < p^2 - 1)$ is the principal cluster of order 2 of this Stirling array, followed by a zero hole of order 2, and clusters headed by -1 and 1 respectively in row $n = p^2$.

The structure of the Stirling array mod p (kind one) can be established analogously to the binomial structure, consisting of layers of clusters alternating with zero holes, with the difference that the cells (clusters) have bases in rows of form $n = ap-2$; and a widening sheaf of zero-entries borders the array from the left.

To obtain a formula analogous to that of Lucas for evaluating $S(n,r) \bmod p$, we transform the array, to obtain a new array without the anomaly of the zero-border on the left.

The formula of transformation is

$$r' = r - \left[\frac{n}{p}\right] + \left[\frac{r - \left[\frac{n}{p}\right]}{p-1}\right] \tag{19}$$

while n remains unchanged, provided that $n \neq ap - 1$. (The symbol [ ] stands for integer part.)

This transforms the Stirling array into one which is more like the binomial array, except that the rows of form $n = ap - 1$ are omitted (these must be considered separately), and gaps appear between the cells, but the r' values corresponding to these do not belong to the range of formula (19). Figure 4 illustrates the situation.

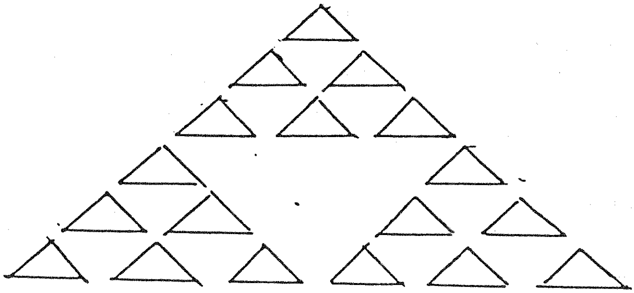

Figure 4

This transformation, together with (18) (adapted also for heads of clusters) provides the means of calculating $S(n,r) \pmod p$ in terms of binomial coefficients and the entries of the principal cell. The similarity principle applies as for the cells and clusters of the binomial array, and using the same "nesting" procedure we arrive to the following theorem.

Theorem. Let $S(n,r)$ be a Stirling number of the second kind and p an odd prime.

Let

$$r' = r - \left[\frac{n}{p}\right] + \left[\frac{r - \left[\frac{n}{p}\right]}{p-1}\right]$$

and let

$$n = a_m p^m + \ldots + a_1 p + a_0 \quad \text{and} \quad r' = b_m p^m + \ldots + b_1 p + b_0.$$

If $n \not\equiv -1 \pmod p$ then

$$S(n,r) \equiv (-1)^{a_m+b_m} \binom{a_m}{b_m} \ldots (-1)^{a_1+b_1} \binom{a_1}{b_1} S(a_0, b_0) \pmod p.$$

This can be written in the form

$$S(n,r) \equiv (-1)^{N+R} \binom{N}{R} S(n_0, r_0) \pmod p \tag{20}$$

where

$$n = Np + n_0 \;\; (0 \le n_0 < p) \quad \text{and} \quad r' = Rp + r_0 \;\; (0 \le r_0 < p).$$

If $n \equiv -1 \pmod p$ then

$$\overline{S(n,r)} = \overline{S(n+1, r+1)}.$$

Stirling numbers, kind 2

For completeness, tables 8 and 9 are attached. These show arrays of Stirling numbers modulo 3 and modulo 5 respectively. These were treated in detail in [6] and [7].

Stirling numbers, kind 2 are defined by the recursion

$$s(n,r) = s(n-1, r-1) + rs(n-1, r)$$

$$s(1,1) = 1;$$

also

$$s(n,r) = 0,$$

when $n = 0$, or $r = 0$ or $r > n$. They have a combinatorial meaning, representing the number of ways in which a set of n distinguishable elements can be placed into r indistinguishable boxes.

Again the arrays modulo p show a cell – zero hole – cluster structure, but with "inclined" rows; Table 8 shows an extended array, while the numerical values of entries are clearer in Table 9.

Only the results are given here. For obtaining a Lucas-type expression for $s(n,r)$, a "straightening" transformation is carried out, defining for $s(n,r)$ the value

$$n' = \left[\frac{np - p\left[\frac{r}{p}\right] - 1}{p-1}\right]$$

and finding the expansions

$$\begin{aligned} n' &= a_m p^m + \ldots + a_1 p + a_0 \\ r &= b_m p^m + \ldots + b_1 p + b_0. \end{aligned}$$

Then

$$\overline{s(n,r)} = \binom{a_m}{b_m} \cdots \binom{a_1}{b_1} \overline{s(a_0, b_0)}.$$

Analogously to (20), we may write

$$s(n,r) \equiv \binom{N}{R} s(n_0, r_0) \pmod p$$

where

$$n' = Np + n_0, \quad r = Rp + r_0,$$

provided that $p \not| r$ and p is an odd prime.

Formulae for the exceptional cases are also calculated and are given in [6].

Other arrays with different recursion formulae have been treated elsewhere, see [6] for Gaussian binomials, and the paper by Sved-Clarke about king's paths on the infinite chessboard [10]. There are still many open problems. Arrays of binomials modulo higher powers of primes were investigated in [8] and [9], but not for more general recursion formulae.

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BINOMIAL COEFFICIENTS MODULO 2

```
 0 1
 1 1 1
 2 1 . 1
 3 1 1 1 1
 4 1 . . . 1
 5 1 1 . . 1 1
 6 1 . 1 . 1 . 1
 7 1 1 1 1 1 1 1 1
 8 1 . . . . . . . 1
 9 1 1 . . . . . . 1 1
10 1 . 1 . . . . . 1 . 1
11 1 1 1 1 . . . . 1 1 1 1
12 1 . . . 1 . . . 1 . . . 1
13 1 1 . . 1 1 . . 1 1 . . 1 1
14 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1
15 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
16 1 . . . . . . . . . . . . . . . 1
17 1 1 . . . . . . . . . . . . . . 1 1
18 1 . 1 . . . . . . . . . . . . . 1 . 1
19 1 1 1 1 . . . . . . . . . . . . 1 1 1 1
20 1 . . . 1 . . . . . . . . . . . 1 . . . 1
21 1 1 . . 1 1 . . . . . . . . . . 1 1 . . 1 1
22 1 . 1 . 1 . 1 . . . . . . . . . 1 . 1 . 1 . 1
23 1 1 1 1 1 1 1 1 . . . . . . . . 1 1 1 1 1 1 1 1
24 1 . . . . . . . 1 . . . . . . . 1 . . . . . . . 1
25 1 1 . . . . . . 1 1 . . . . . . 1 1 . . . . . . 1 1
26 1 . 1 . . . . . 1 . 1 . . . . . 1 . 1 . . . . . 1 . 1
27 1 1 1 1 . . . . 1 1 1 1 . . . . 1 1 1 1 . . . . 1 1 1 1
28 1 . . . 1 . . . 1 . . . 1 . . . 1 . . . 1 . . . 1 . . . 1
29 1 1 . . 1 1 . . 1 1 . . 1 1 . . 1 1 . . 1 1 . . 1 1 . . 1 1
30 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1 . 1
31 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
32 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1
33 1 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1 1
34 1 . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1 . 1
35 1 1 1 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1 1 1 1
36 1 . . . 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . 1 . . . 1
37 1 1 . . 1 1 . . . . . . . . . . . . . . . . . . . . . . . . . . 1 1 . . 1 1
38 1 . 1 . 1 . 1 . . . . . . . . . . . . . . . . . . . . . . . . . 1 . 1 . 1 . 1
39 1 1 1 1 1 1 1 1 . . . . . . . . . . . . . . . . . . . . . . . . 1 1 1 1 1 1 1 1
40 1 . . . . . . . 1 . . . . . . . . . . . . . . . . . . . . . . . 1 . . . . . . . 1
41 1 1 . . . . . . 1 1 . . . . . . . . . . . . . . . . . . . . . . 1 1 . . . . . . 1 1
421 . 1 . . . . . . 1 . 1 . . . . . . . . . . . . . . . . . . . . . 1 . 1 . . . . . 1 . .
```

TABLE 2

BINOMIAL COEFFICIENTS MODULO 5

0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 1 4 1
5	1 1
6	1 1 . . . 1 1
7	1 2 1 . . 1 2 1
8	1 3 3 1 . 1 3 3 1
9	1 4 1 4 1 1 4 1 4 1
10	1 . . . . 2 1
11	1 1 . . . 2 2 . . . 1 1
12	1 2 1 . . 2 4 2 . . 1 2 1
13	1 3 3 1 . 2 1 1 2 . 1 3 3 1
14	1 4 1 4 1 2 3 2 3 2 1 4 1 4 1
15	1 . . . . 3 . . . . 3 1
16	1 1 . . . 3 3 . . . 3 3 . . . 1 1
17	1 2 1 . . 3 1 3 . . 3 1 3 . . 1 2 1
18	1 3 3 1 . 3 4 4 3 . 3 4 4 3 . 1 3 3 1
19	1 4 1 4 1 3 2 3 2 3 3 2 3 2 3 1 4 1 4 1
20	1 . . . . 4 . . . . 1 . . . . 4 1
21	1 1 . . . 4 4 . . . 1 1 . . . 4 4 . . . 1 1
22	1 2 1 . . 4 3 4 . . 1 2 1 . . 4 3 4 . . 1 2 1
23	1 3 3 1 . 4 2 2 4 . 1 3 3 1 . 4 2 2 4 . 1 3 3 1
24	1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1
25	1 1
26	1 1 . . . . 1 1
27	1 2 1 . . . . 1 2 1
28	1 3 3 1 . . . . 1 3 3 1
29	1 4 1 4 1 . . . . 1 4 1 4 1
30	1 . . . . 1 . . . . 1 1
31	1 1 . . . 1 1 . . . . 1 1 . . . 1 1
32	1 2 1 . . 1 2 1 . . . . 1 2 1 . . 1 2 1
33	1 3 3 1 . 1 3 3 1 . . . . 1 3 3 1 . 1 3 3 1
34	1 4 1 4 1 1 4 1 4 1 . . . . 1 4 1 4 1 1 4 1 4 1
35	1 . . . . 2 . . . . 1 . . . . 1 . . . . 2 1
36	1 1 . . . 2 2 . . . 1 1 . . . . 1 1 . . . 2 2 . . . 1 1
37	1 2 1 . . 2 4 2 . . 1 2 1 . . . . 1 2 1 . . 2 4 2 . . 1 2 1
38	1 3 3 1 . 2 1 1 2 . 1 3 3 1 . . . . 1 3 3 1 . 2 1 1 2 . 1 3 3 1
39	1 4 1 4 1 2 3 2 3 2 1 4 1 4 1 . . . . 1 4 1 4 1 2 3 2 3 2 1 4 1 4 1
40	1 . . . . 3 . . . . 3 . . . . 1 . . . . 1 . . . . 3 . . . . 3 1
41	1 1 . . . 3 3 . . . 3 3 . . . 1 1 . . . . 1 1 . . . 3 3 . . . 3 3 . . . 1 1
42	1 2 1 . . 3 1 3 . . 3 1 3 . . 1 2 1 . . . . 1 2 1 . . 3 1 3 . . 3 1 3 . . 1 2 1
43	1 3 3 1 . 3 4 4 3 . 3 4 4 3 . 1 3 3 1 . . . . . . 1 3 3 1 . 3 4 4 3 . 3 4 4 3 . 1 3 3 1
44	1 4 1 4 1 3 2 3 2 3 3 2 3 2 3 1 4 1 4 1 . . . . . 1 4 1 4 1 3 2 3 2 3 3 2 3 2 3 1 4 1 4 1

TABLE 3

LINEAR RECURSION MODULO 5

```
 0  1
 1  2 3
 2  4 2 4
 3  3 1 4 2
 4  1 1 1 1 1
 5  2 . . . . 3
 6  4 1 . . . 1 4
 7  3 4 3 . . 2 1 2
 8  1 2 3 4 . 4 3 2 1
 9  2 2 2 2 2 3 3 3 3 3
10  4 . . . . 2 . . . . 4
11  3 2 . . . 4 1 . . . 3 2
12  1 3 1 . . 3 4 3 . . 1 3 1
13  2 4 1 3 . 1 2 3 4 . 2 4 1 3
14  4 4 4 4 4 2 2 2 2 2 4 4 4 4 4
15  3 . . . . 1 . . . . 4 . . . . 2
16  1 4 . . . 2 3 . . . 3 2 . . . 4 1
17  2 1 2 . . 4 2 4 . . 1 3 1 . . 3 4 3
18  4 3 2 1 . 3 1 4 2 . 2 4 1 3 . 1 2 3 4
19  3 3 3 3 3 1 1 1 1 1 4 4 4 4 4 2 2 2 2 2
20  1 . . . . 1 . . . . 1 . . . . 1 . . . . 1
21  2 3 . . . 2 3 . . . 2 3 . . . 2 3 . . . 2 3
22  4 2 4 . . 4 2 4 . . 4 2 4 . . 4 2 4 . . 4 2 4
23  3 1 4 2 . 3 1 4 2 . 3 1 4 2 . 3 1 4 2 . 3 1 4 2
24  1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
25  2 . . . . . . . . . . . . . . . . . . . . . . . . 3
26  4 1 . . . . . . . . . . . . . . . . . . . . . . . 1 4
27  3 4 3 . . . . . . . . . . . . . . . . . . . . . . 2 1 2
28  1 2 3 4 . . . . . . . . . . . . . . . . . . . . . 4 3 2 1
29  2 2 2 2 2 . . . . . . . . . . . . . . . . . . . . 3 3 3 3 3
30  4 . . . . 1 . . . . . . . . . . . . . . . . . . . 1 . . . . 4
31  3 2 . . . 2 3 . . . . . . . . . . . . . . . . . . 2 3 . . . 3 2
32  1 3 1 . . 4 2 4 . . . . . . . . . . . . . . . . . 4 2 4 . . 1 3 1
33  2 4 1 3 . 3 1 4 2 . . . . . . . . . . . . . . . . 3 1 4 2 . 2 4 1 3
34  4 4 4 4 4 1 1 1 1 1 . . . . . . . . . . . . . . . 1 1 1 1 1 4 4 4 4 4
35  3 . . . . 4 . . . . 3 . . . . . . . . . . . . . . 2 . . . . 1 . . . . 2
36  1 4 . . . 3 2 . . . 1 4 . . . . . . . . . . . . . 4 1 . . . 2 3 . . . 4 1
37  2 1 2 . . 1 3 1 . . 2 1 2 . . . . . . . . . . . . 3 4 3 . . 4 2 4 . . 3 4 3
38  4 3 2 1 . 2 4 1 3 . 4 3 2 1 . . . . . . . . . . . 1 2 3 4 . 3 1 4 2 . 1 2 3 4
39  3 3 3 3 3 4 4 4 4 4 3 3 3 3 3 . . . . . . . . . . 2 2 2 2 2 1 1 1 1 1 2 2 2 2 2
40  1 . . . . 2 . . . . 3 . . . . 4 . . . . . . . . . 4 . . . . 3 . . . . 2 . . . . 1
41  2 3 . . . 4 1 . . . 1 4 . . . 3 2 . . . . . . . . 3 2 . . . 1 4 . . . 4 1 . . . 2 3
42  4 2 4 . . 3 4 3 . . 2 1 2 . . 1 3 1 . . . . . . . 1 3 1 . . 2 1 2 . . 3 4 3 . . 4 2 4
43  3 1 4 2 . 1 2 3 4 . 4 3 2 1 . 2 4 1 3 . . . . . . 2 4 1 3 . 4 3 2 1 . 1 2 3 4 . 3 1 4 2
44  1 1 1 1 1 2 2 2 2 2 3 3 3 3 3 4 4 4 4 4 . . . . . 4 4 4 4 4 3 3 3 3 3 2 2 2 2 2 1 1 1 1 1
```

TABLE 4

ARRAY SHOWING NO.OF WALKS MODULO 5 B,C= 1 1
A, B, C,D ARE 0 1 1 0

```
 0 1111111111111111111111111111111111111111111111111111111111111111111111111111
 1 1234.1234.1234.1234.1234.1234.1234.1234.1234.1234.1234.1234.1234.1234.1234.1
 2 131..131..131..131..131..131..131..131..131..131..131..131..131..131..131..1
 3 14...14...14...14...14...14...14...14...14...14...14...14...14...14...14...1
 4 1....1....1....1....1....1....1....1....1....1....1....1....1....1....1....1
 5 11111222223333344444.....11111222223333344444.....11111222223333344444.....1
 6 1234.2413.3142.4321......1234.2413.3142.4321......1234.2413.3142.4321......1
 7 131..212..343..424.......131..212..343..424.......131..212..343..424.......1
 8 14...23...32...41........14...23...32...41........14...23...32...41........1
 9 1....2....3....4.........1....2....3....4.........1....2....3....4.........1
10 111113333311111..........111113333311111..........111113333311111..........1
11 1234.3142.1234...........1234.3142.1234...........1234.3142.1234...........1
12 131..343..131............131..343..131............131..343..131............1
13 14...32...14.............14...32...14.............14...32...14.............1
14 1....3....1..............1....3....1..............1....3....1..............1
15 1111144444...............1111144444...............1111144444...............1
16 1234.4321................1234.4321................1234.4321................1
17 131..424.................131..424.................131..424.................1
18 14...41..................14...41..................14...41..................1
19 1....4...................1....4...................1....4...................1
20 11111....................11111....................11111....................1
21 1234.....................1234.....................1234.....................1
22 131......................131......................131......................1
23 14.......................14.......................14.......................1
24 1........................1........................1........................1
25 1111111111111111111111111222222222222222222222222233333333333333333333333334
26 1234.1234.1234.1234.1234.2413.2413.2413.2413.2413.3142.3142.3142.3142.3142.4
27 131..131..131..131..131..212..212..212..212..212..343..343..343..343..343..4
28 14...14...14...14...14...23...23...23...23...23...32...32...32...32...32...4
29 1....1....1....1....1....2....2....2....2....2....3....3....3....3....3....4
30 11111222223333344444.....22222444441111133333.....33333111114444422222.....4
31 1234.2413.3142.4321......2413.4321.1234.3142......3142.1234.4321.2413......4
32 131..212..343..424.......212..424..131..343.......343..131..424..212.......4
33 14...23...32...41........23...41...14...32........32...14...41...23........4
34 1....2....3....4.........2....4....1....3.........3....1....4....2.........4
35 111113333311111..........222221111122222..........333334444433333..........4
36 1234.3142.1234...........2413.1234.2413...........3142.4321.3142...........4
37 131..343..131............212..131..212............343..424..343............4
38 14...32...14.............23...14...23.............32...41...32.............4
39 1....3....1..............2....1....2..............3....4....3..............4
40 1111144444...............2222233333...............3333322222...............4
41 1234.4321................2413.3142................3142.2413................4
42 131..424.................212..343.................343..212.................4
43 14...41..................23...32..................32...23..................4
44 1....4...................2....3...................3....2...................4
45 11111....................22222....................33333....................4
46 1234.....................2413.....................3142.....................4
47 131......................212......................343......................4
48 14.......................23.......................32.......................4
49 1........................2........................3........................4
50 111111111111111111111111133333333333333333333333331111111111111111111111111.
51 1234.1234.1234.1234.1234.3142.3142.3142.3142.3142.1234.1234.1234.1234.1234..
52 131..131..131..131..131..343..343..343..343..343..131..131..131..131..131...
53 14...14...14...14...14...32...32...32...32...32...14...14...14...14...14....
54 1....1....1....1....1....3....3....3....3....3....1....1....1....1....1.....
55 11111222223333344444.....33333111114444422222.....11111222223333344444......
56 1234.2413.3142.4321......3142.1234.4321.2413......1234.2413.3142.4321.......
57 131..212..343..424.......343..131..424..212.......131..212..343..424........
58 14...23...32...41........32...14...41...23........14...23...32...41.........
59 1....2....3....4.........3....1....4....2.........1....2....3....4..........
60 111113333311111..........333334444433333..........111113333311111...........
```

TABLE 5 ARRAY SHOWING NO.OF WALKS MODULO 5 B,C= 2 3

A, B, C,D ARE 0 2 3 0

```
 0 1342134213421342134213421342134213421342134213421342134213421342134213421342
 1 2241.1123.3314.4432.2241.1123.3314.4432.2241.1123.3314.4432.2241.1123.3314.4
 2 411..233..144..322..411..233..144..322..411..233..144..322..411..233..144..3
 3 31...43...24...12...31...43...24...12...31...43...24...12...31...43...24...1
 4 1....3....4....2....1....3....4....2....1....3....4....2....1....3....4....2
 5 21342213424213413421.....13421134212134234213.....34213342131342142134.....4
 6 4432.4432.3314.2241......2241.2241.4432.1123......1123.1123.2241.3314......3
 7 322..322..144..411.......411..411..322..233.......233..233..411..144.......1
 8 12...12...24...31........31...31...12...43........43...43...31...24........2
 9 2....2....4....1.........1....1....2....3.........3....3....1....4.........4
10 421341342113421..........213423421334213..........134214213442134..........3
11 3314.2241.2241...........4432.1123.1123...........2241.3314.3314...........1
12 144..411..411............322..233..233............411..144..144............2
13 24...31...31.............12...43...43.............31...24...24.............4
14 4....1....1..............2....3....3..............1....4....4..............3
15 3421313421...............4213434213...............2134242134...............1
16 1123.2241................3314.1123................4432.3314................2
17 233..411.................144..233.................322..144.................4
18 43...31..................24...43..................12...24..................3
19 3....1...................4....3...................2....4...................1
20 13421....................34213....................42134....................2
21 2241.....................1123.....................3314.....................4
22 411......................233......................144......................3
23 31.......................43.......................24.......................1
24 1........................3........................4........................2
25 2134213421342134213421342213421342134213421342421342134213421342134213421341
26 4432.2241.1123.3314.4432.4432.2241.1123.3314.4432.3314.4432.2241.1123.3314.2
27 322..411..233..144..322..322..411..233..144..322..144..322..411..233..144..4
28 12...31...43...24...12...12...31...43...24...12...24...12...31...43...24...3
29 2....1....3....4....2....2....1....3....4....2....4....2....1....3....4....1
30 42134421343421321342.....42134421343421321342.....34213342131342142134.....2
31 3314.3314.1123.4432......3314.3314.1123.4432......1123.1123.2241.3314......4
32 144..144..233..322.......144..144..233..322.......233..233..411..144.......3
33 24...24...43...12........24...24...43...12........43...43...31...24........1
34 4....4....3....2.........4....4....3....2.........3....3....1....4.........2
35 342132134221342..........342132134221342..........134214213442134..........4
36 1123.4432.4432...........1123.4432.4432...........2241.3314.3314...........3
37 233..322..322............233..322..322............411..144..144............1
38 43...12...12.............43...12...12.............31...24...24.............2
39 3....2....2..............3....2....2..............1....4....4..............4
40 1342121342...............1342121342...............2134242134...............3
41 2241.4432................2241.4432................4432.3314................1
42 411..322.................411..322.................322..144.................2
43 31...12..................31...12..................12...24..................4
44 1....2...................1....2...................2....4...................3
45 21342....................21342....................42134....................1
46 4432.....................4432.....................3314.....................2
47 322......................322......................144......................4
48 12.......................12.......................24.......................3
49 2........................2........................4........................1
50 4213421342134213421342134134213421342134213421342113421342134213421342134213421.
51 3314.4432.2241.1123.3314.2241.1123.3314.4432.2241.2241.1123.3314.4432.2241..
52 144..322..411..233..144..411..233..144..322..411..411..233..144..322..411...
53 24...12...31...43...24...31...43...24...12...31...31...43...24...12...31....
54 4....2....1....3....4....1....3....4....2....1....1....3....4....2....1.....
55 34213342131342142134.....21342213424213413421.....21342213424213413421......
56 1123.1123.2241.3314......4432.4432.3314.2241......4432.4432.3314.2241.......
57 233..233..411..144.......322..322..144..411.......322..322..144..411........
58 43...43...31...24........12...12...24...31........12...12...24...31.........
59 3....3....1....4.........2....2....4....1.........2....2....4....1..........
60 134214213442134..........421341342113421..........421341342113421...........
```

TABLE 6

RESTRICTED WALK MODULO 5
A, B, C,D ARE 0 1 1 0

```
 0 1
 1 11
 2 122
 3 13..
 4 14444
 5 1.4322
 6 11.3.22
 7 122..244
 8 13...21..
 9 1444412222
10 1.4323.2411
11 11.3.33.4.11
12 122..311..122
13 13...34...13..
14 14444211112....
15 1.4324.123......
16 11.3.44.2........
17 122..433..........
18 13...42............
19 144443..............
20 1.432................
21 11.3..................
22 122....................
23 13......................
24 1444444444444444444444444
25 1.4321.4321.4321.4321.4322
26 11.3.11.3.11.3.11.3.11.3.22
27 122..122..122..122..122..244
28 13...13...13...13...13...21..
29 14444.333342222311112....23333
30 1.4322.3143.2414.123.....2.3144
31 11.3.22.1.33.4.44.2......22.1.44
32 122..244..311..433.......244..433
33 13...21...34...42........21...42..
34 1444412222.44443.........2333324444
35 1.4323.2411.432..........2.3141.4322
36 11.3.33.4.11.3...........22.1.11.3.22
37 122..311..122............244..122..244
38 13...34...13.............21...13...21..
39 14444211112..............23333422224....
40 1.4324.123...............2.3143.241......
41 11.3.44.2................22.1.33.4........
42 122..433.................244..311..........
43 13...42..................21...34............
44 144443...................233331..............
45 1.432....................2.314................
46 11.3.....................22.1..................
47 122......................244....................
48 13.......................21......................
49 14444444444444444444444441222222222222222222222222
50 1.4321.4321.4321.4321.4323.2413.2413.2413.2413.2411
51 11.3.11.3.11.3.11.3.11.3.33.4.33.4.33.4.33.4.33.4.11
52 122..122..122..122..122..311..311..311..311..311..122
53 13...13...13...13...13...34...34...34...34...34...13..
54 14444.333342222311112....32222.444421111433331....14444
55 1.4322.3143.2414.123.....3.2411.4324.1232.314.....1.4322
56 11.3.22.1.33.4.44.2......33.4.11.3.44.2.22.1......11.3.22
57 122..244..311..433.......311..122..433..244.......122..244
58 13...21...34...42........34...13...42...21........13...21..
59 1444412222.44443.........3222231111.22224.........1444412222
60 1.4323.2411.432..........3.2414.1233.241..........1.4323.2411
```

TABLE 7

STIRLING NUMBERS, KIND ONE, MODULO 5

```
0   1
1   4 1
2   2 2 1
3   4 1 4 1
4   4 . . . 1
5   . 4 . . . 1
6   . 1 4 . . 4 1
7   . 3 3 4 . 2 2 1
8   . 1 4 1 4 4 1 4 1
9   . 1 . . . 3 . . . 1
10  . . 1 . . . 3 . . . 1
11  . . 4 1 . . 2 3 . . 4 1
12  . . 2 2 1 . 1 1 3 . 2 2 1
13  . . 4 1 4 1 2 3 2 3 4 1 4 1
14  . . 4 . . . 3 . . . 2 . . . 1
15  . . . 4 . . . 3 . . . 2 . . . 1
16  . . . 1 4 . . 2 3 . . 3 2 . . 4 1
17  . . . 3 3 4 . 1 1 3 . 4 4 2 . 2 2 1
18  . . . 1 4 1 4 2 3 2 3 3 2 3 2 4 1 4 1
19  . . . 1 . . . 1 . . . 1 . . . 1 . . . 1
20  . . . . 1 . . . 1 . . . 1 . . . 1 . . . 1
21  . . . . 4 1 . . 4 1 . . 4 1 . . 4 1 . . 4 1
22  . . . . 2 2 1 . 2 2 1 . 2 2 1 . 2 2 1 . 2 2 1
23  . . . . 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1
24  . . . . 4 . . . . . . . . . . . . . . . . . . . 1
25  . . . . . 4 . . . . . . . . . . . . . . . . . . . 1
26  . . . . . 1 4 . . . . . . . . . . . . . . . . . . 4 1
27  . . . . . 3 3 4 . . . . . . . . . . . . . . . . . 2 2 1
28  . . . . . 1 4 1 4 . . . . . . . . . . . . . . . . 4 1 4 1
29  . . . . . 1 . . . 4 . . . . . . . . . . . . . . . 4 . . . 1
30  . . . . . . 1 . . . 4 . . . . . . . . . . . . . . . 4 . . . 1
31  . . . . . . 4 1 . . 1 4 . . . . . . . . . . . . . . 1 4 . . 4 1
32  . . . . . . 2 2 1 . 3 3 4 . . . . . . . . . . . . . 3 3 4 . 2 2 1
33  . . . . . . 4 1 4 1 1 4 1 4 . . . . . . . . . . . . 1 4 1 4 4 1 4 1
34  . . . . . . 4 . . . 2 . . . 4 . . . . . . . . . . . 1 . . . 3 . . . 1
35  . . . . . . . 4 . . . 2 . . . 4 . . . . . . . . . . . 1 . . . 3 . . . 1
36  . . . . . . . 1 4 . . 3 2 . . 1 4 . . . . . . . . . . 4 1 . . 2 3 . . 4 1
37  . . . . . . . 3 3 4 . 4 4 2 . 3 3 4 . . . . . . . . . 2 2 1 . 1 1 3 . 2 2 1
38  . . . . . . . 1 4 1 4 3 2 3 2 1 4 1 4 . . . . . . . . 4 1 4 1 2 3 2 3 4 1 4 1
39  . . . . . . . 1 . . . 2 . . . 3 . . . 4 . . . . . . . 4 . . . 3 . . . 2 . . . 1
40  . . . . . . . . 1 . . . 2 . . . 3 . . . 4 . . . . . . . 4 . . . 3 . . . 2 . . . 1
41  . . . . . . . . 4 1 . . 3 2 . . 2 3 . . 1 4 . . . . . . 1 4 . . 2 3 . . 3 2 . . 4 1
42  . . . . . . . . 2 2 1 . 4 4 2 . 1 1 3 . 3 3 4 . . . . . 3 3 4 . 1 1 3 . 4 4 2 . 2 2 1
43  . . . . . . . . 4 1 4 1 3 2 3 2 2 3 2 3 1 4 1 4 . . . . 1 4 1 4 2 3 2 3 3 2 3 2 4 1 4 1
44  . . . . . . . . 4 . . . 4 . . . 4 . . . 4 . . . 4 . . . 1 . . . 1 . . . 1 . . . 1 . . . .
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TABLE 8

STIRLING NUMBERS, KIND TWO, MODULO 5

1	1
2	1 1
3	1 3 1
4	1 2 1 1
5	1 . . . 1
6	1 1 . . . 1
7	1 3 1 . . 1 1
8	1 2 1 1 . 1 3 1
9	1 . . . 1 1 2 1 1
10	1 1 . . . 2 . . . 1
11	1 3 1 . . 2 2 . . . 1
12	1 2 1 1 . 2 1 2 . . 1 1
13	1 . . . 1 2 4 2 2 . 1 3 1
14	1 1 . . . 3 . . . 2 1 2 1 1
15	1 3 1 . . 3 3 . . . 3 . . . 1
16	1 2 1 1 . 3 4 3 . . 3 3 . . . 1
17	1 . . . 1 3 1 3 3 . 3 4 3 . . 1 1
18	1 1 . . . 4 . . . 3 3 1 3 3 . 1 3 1
19	1 3 1 . . 4 4 . . . 1 . . . 3 1 2 1 1
20	1 2 1 1 . 4 2 4 . . 1 1 . . . 4 . . . 1
21	1 . . . 1 4 3 4 4 . 1 3 1 . . 4 4 . . . 1
22	1 1 . . . . 4 1 2 1 1 . 4 2 4 . . 1 1
23	1 3 1 . . . . 1 4 3 4 4 . 1 3 1
24	1 2 1 1 . . . . 4 1 2 1 1
25	1 . . . 1 1
26	1 1 . . . 1 1
27	1 3 1 . . 1 1 . . . . 1 1
28	1 2 1 1 . 1 3 1 . . . . 1 3 1
29	1 . . . 1 1 2 1 1 . . . . 1 2 1 1
30	1 1 . . . 2 . . . 1 . . . . 1 . . . 1
31	1 3 1 . . 2 2 . . . 1 . . . . 1 1 . . . 1
32	1 2 1 1 . 2 1 2 . . 1 1 . . . . 1 3 1 . . 1 1
33	1 . . . 1 2 4 2 2 . 1 3 1 . . . . 1 2 1 1 . 1 3 1
34	1 1 . . . 3 . . . 2 1 2 1 1 . . . . 1 . . . 1 1 2 1 1
35	1 3 1 . . 3 3 . . . 3 . . . 1 . . . . 1 1 . . . 2 . . . 1
36	1 2 1 1 . 3 4 3 . . 3 3 . . . 1 . . . . 1 3 1 . . 2 2 . . . 1
37	1 . . . 1 3 1 3 3 . 3 4 3 . . 1 1 . . . . 1 2 1 1 . 2 1 2 . . 1 1
38	1 1 . . . 4 . . . 3 3 1 3 3 . 1 3 1 . . . . 1 . . . 1 2 4 2 2 . 1 3 1
39	1 3 1 . . 4 4 . . . 1 . . . 3 1 2 1 1 . . . . . . 1 1 . . . 3 . . . 2 1 2 1 1
40	1 2 1 1 . 4 2 4 . . 1 1 . . . 4 . . . 1 . . . . . 1 3 1 . . 3 3 . . . 3 . . . 1
41	1 . . . 1 4 3 4 4 . 1 3 1 . . 4 4 . . . 1 . . . . 1 2 1 1 . 3 4 3 . . 3 3 . . . 1
42	1 1 . . . . 4 1 2 1 1 . 4 2 4 . . 1 1 . . . 1 . . . 1 3 1 3 3 . 3 4 3 . . 1 1
43	1 3 1 . . . . 1 4 3 4 4 . 1 3 1 . . 1 1 . . . 4 . . . 3 3 1 3 3 . 1 3 1
44	1 2 1 1 . . . . 4 1 2 1 1 . 1 3 1 . . 4 4 . . . 1 . . . 3 1 2 1 1
45	1 . . . 1 . . . . 1 1 2 1 1 . 4 2 4 . . 1 1 . . . 4 . . . 1

TABLE 9

STIRLING NUMBERS, KIND TWO, MODULO 3

1	1
2	1 1
3	1 . 1
4	1 1 . 1
5	1 . 1 1 1
6	1 1 . 2 . 1
7	1 . 1 2 2 . 1
8	1 1 . . . 2 1 1
9	1 . 1 1
10	1 1 . 1 1
11	1 . 1 1 1 . . . . 1 1
12	1 1 . 2 . 1 . . . 1 . 1
13	1 . 1 2 2 . 1 . . 1 1 . 1
14	1 1 . . . 2 1 1 . 1 . 1 1 1
15	1 . 1 . . . . . 1 1 1 . 2 . 1
16	1 1 . 1 . . . . . 2 . 1 2 2 . 1
17	1 . 1 1 1 . . . . 2 2 . . . 2 1 1
18	1 1 . 2 . 1 . . . 2 . 2 1
19	1 . 1 2 2 . 1 . . 2 2 . 2 1
20	1 1 . . . 2 1 1 . 2 . 2 2 2 . . . . 1 1
21	1 . 1 . . . . . 1 2 2 . 1 . 2 . . . 1 . 1
22	1 1 . 1 . . . . 2 1 1 . 2 . . 1 1 . 1
23	1 . 1 1 1 . . . . 1 2 2 . 1 . 1 1 1
24	1 1 . 2 . 1 . . . . 2 1 1 . 2 . 1
25	1 . 1 2 2 . 1 . . . . 1 2 2 . 1
26	1 1 . . . 2 1 1 . . . . 2 1 1
27	1 . 1 . . . . . 1 1
28	1 1 . 1 . . . . . 1 1
29	1 . 1 1 1 . . . . 1 1 . . . . 1 1
30	1 1 . 2 . 1 . . . 1 . 1 . . . . 1 . 1
31	1 . 1 2 2 . 1 . . 1 1 . 1 . . . . 1 1 . 1
32	1 1 . . . 2 1 1 . 1 . 1 1 1 . . . . 1 . 1 1 1
33	1 . 1 . . . . . 1 1 1 . 2 . 1 . . . . 1 1 . 2 . 1
34	1 1 . 1 . . . . . 2 . 1 2 2 . 1 . . . . 1 . 1 2 2 . 1
35	1 . 1 1 1 . . . . 2 2 . . . 2 1 1 . . . . 1 1 . . . 2 1 1
36	1 1 . 2 . 1 . . . 2 . 2 . . . . . 1 . . . . 1 . 1 1
37	1 . 1 2 2 . 1 . . 2 2 . 2 . . . . . 1 . . . . 1 1 . 1 1
38	1 1 . . . 2 1 1 . 2 . 2 2 2 . . . . 1 1 . . . . 1 . 1 1 1 . . . . 1 1
39	1 . 1 . . . . . 1 2 2 . 1 . 2 . . . 1 . 1 . . . . . . 1 1 . 2 . 1 . . . 1 . 1
40	1 1 . 1 . . . . 2 1 1 . 2 . . 1 1 . 1 . . . . . 1 . 1 2 2 . 1 . . 1 1 . 1
41	1 . 1 1 1 . . . . 1 2 2 . 1 . 1 1 1 . . . . 1 1 . . . 2 1 1 . 1 . 1 1 1
42	1 1 . 2 . 1 . . . . 2 1 1 . 2 . 1 . . . 1 . 1 . . . . . 1 1 1 . 2 . 1
43	1 . 1 2 2 . 1 . . . . 1 2 2 . 1 . . 1 1 . 1 . . . . . 2 . 1 2 2 . 1
44	1 1 . . . 2 1 1 . . . . 2 1 1 . 1 . 1 1 1 . . . . 2 2 . . . 2 1 1
45	1 . 1 . . . . . 1 . . . . 1 1 1 . 2 . 1 . . . 2 . 2 1
46	1 1 . 1 . . . . . 1 . . . . 2 . 1 2 2 . 1 . . 2 2 . 2 1
47	1 . 1 1 1 . . . . 1 1 . . . . 2 2 . . . 2 1 1 . 2 . 2 2 2 . . . . 1 1
48	1 1 . 2 . 1 . . . 1 . 1 . . . . 2 . 2 . . . . . 1 2 2 . 1 . 2 . . . 1 . 1
49	1 . 1 2 2 . 1 . . 1 1 . 1 . . . . 2 2 . 2 . . . . 2 1 1 . 2 . . 1 1 . 1
50	1 1 . . . 2 1 1 . 1 . 1 1 1 . . . . 2 . 2 2 2 . . . . 1 2 2 . 1 . 1 1 1
51	1 . 1 . . . . . 1 1 1 . 2 . 1 . . . . 2 2 . 1 . 2 . . . . 2 1 1 . 2 . 1
52	1 1 . 1 . . . . . 2 . 1 2 2 . 1 . . . . 2 . 2 1 1 . 2 . . . . 1 2 2 . 1
53	1 . 1 1 1 . . . . 2 2 . . . 2 1 1 . . . . 2 2 . . . 1 2 2 . . . . 2 1 1
54	1 1 . 2 . 1 . . . 2 . 2 . . . . . 1 . . . . 2 . 2 . . . . . 2 1
55	1 . 1 2 2 . 1 . . 2 2 . 2 . . . . . 1 . . . . 2 2 . 2 . . . . . 2 1
56	1 1 . . . 2 1 1 . 2 . 2 2 2 . . . . 1 1 . . . . 2 . 2 2 2 . . . . 2 2 . . . . 1 1
57	1 . 1 . . . . . 1 2 2 . 1 . 2 . . . 1 . 1 . . . . . . 2 2 . 1 . 2 . . . 2 . 2 . . . . 1 . 1
58	1 1 . 1 . . . . 2 1 1 . 2 . . 1 1 . 1 . . . . . 2 . 2 1 1 . 2 . . 2 2 . 2 . . . . 1 1 . 1
59	1 . 1 1 1 . . . . 1 2 2 . 1 . 1 1 1 . . . . 2 2 . . . 1 2 2 . 2 . 2 2 2 . . . . 1 . 1 1
60	1 1 . 2 . 1 . . . . 2 1 1 . 2 . 1 . . . 2 . 2 . . . . . 2 2 2 . 1 . 2 . . . . 1 1 . 2 .
61	1 . 1 2 2 . 1 . . . . 1 2 2 . 1 . . 2 2 . 2 . . . . . 1 . 2 1 1 . 2 . . . . 1 . 1 2 2
62	1 1 . . . 2 1 1 . . . . 2 1 1 . 2 . 2 2 2 . . . . 1 1 . . . 1 2 2 . . . . 1 1 . . . 2
63	1 . 1 . . . . . 1 . . . . 1 2 2 . 1 . 2 . . . 1 . 1 . . . . . 2 . . . . 1 . 1 . . .
64	1 1 . 1 . . . . . 1 . . . . 2 1 1 . 2 . . 1 1 . 1 . . . . . 2 . . . . 1 1 . 1 . . .
65	1 . 1 1 1 . . . . 1 1 . . . . 1 2 2 . 1 . 1 1 1 . . . . 2 2 . . . . 1 . 1 1 1 . .
66	1 1 . 2 . 1 . . . 1 . 1 . . . . 2 1 1 . 2 . 1 . . . 2 . 2 . . . . . . 1 1 . 2 . 1 . .
67	1 . 1 2 2 . 1 . . 1 1 . 1 . . . . 1 2 2 . 1 . . 2 2 . 2 . . . . . 1 . 1 2 2 . 1 .
68	1 1 . . . 2 1 1 . 1 . 1 1 1 . . . . 2 1 1 . 2 . 2 2 2 . . . . 1 1 . . . 2 1 1 .
69	1 . 1 . . . . . 1 1 1 . 2 . 1 . . . . 1 2 2 . 1 . 2 . . . 1 . 1 1
70	1 1 . 1 . . . . . 2 . 1 2 2 . 1 . . . . 2 1 1 . 2 . . 1 1 . 1 2
71	1 . 1 1 1 . . . . 2 2 . . . 2 1 1 . . . . 1 2 2 . 1 . 1 1 1 2
72	1 1 . 2 . 1 . . . 2 . 2 . . . . . 1 . . . . 2 1 1 . 2 . 1 . . . 2 .
73	1 . 1 2 2 . 1 . . 2 2 . 2 . . . . . 1 . . . . 1 2 2 . 1 . . 2 2