Small Non-Isomorphic Repeated Measurements Designs

Sandra J. Pattison[1] and Deborah J. Street[2]

[1] Department of Statistics, The University of Adelaide
[2] Department of Statistics, The University of N.S.W.

Abstract

Over the last few years a number of authors have investigated the structure of optimal repeated measurements designs. Various constructions for such designs have been given. In this paper we consider the construction of non-isomorphic optimal repeated measurements designs when t=2 and 3.

1. Introduction

In a *repeated measurements design* (RMD) there are t treatments, n experimental units and the experiment lasts for p periods. Each experimental unit receives one treatment during each period. Thus the design may be represented as a pxn array containing entries from {1,2,...,t}. Examples of RMDs with t=2, p=4 and n=4 appear in Table 1.

1	1	2	2		1	1	1	2
2	2	1	1		1	2	2	1
1	1	2	2		2	2	2	1
2	2	1	1		2	1	1	1
	(a)					(b)		

Table 1. *Examples of RMDs.*

A RMD is said to be *uniform on units* (or columns) if each treatment appears the same number of times in each column, and to be *uniform on periods* (or rows) if each treatment appears the same number of times in each row. A RMD is said to be *uniform* if it is uniform on both units and periods. Thus, in a uniform RMD, each treatment appears p/t times in each column and n/t times in each row. Hence necessary conditions for the existence of uniform RMDs are $t|p$ and $t|n$. The design (a) in Table 1 is a uniform RMD, whereas design (b) is not uniform on either rows or columns.

Let m_{ij} denote the number of times that treatment i is preceded by treatment j. A

RMD is said to be *balanced* if

$$m_{ij} = (1-\delta_{ij})\,\frac{n(p-1)}{t(t-1)},\quad 1 \le i,j \le t,$$

where δ_{ij} is the Kronecker δ, and to be *strongly balanced* if

$$m_{ij} = \frac{n(p-1)}{t^2},\quad 1 \le i,j \le t.$$

The design (a) in Table 1 is balanced and design (b) is strongly balanced.

The linear models associated with these designs have been given by a number of authors (see, for example, Cheng and Wu (1980), Kunert (1984) and Street (1988)). Cheng and Wu (1980) have shown that one class of optimal designs are the strongly balanced, uniform RMDs and they give a construction for such designs when $n=t^2$ and $p=2t$. Placing two such designs side-by-side gives a strongly balanced, uniform design with $n=2t^2$ and $p=2t$, and placing two of their designs one under the other gives a strongly balanced, uniform RMD with $n=t^2$ and $p=2(2t)$. (These are examples of pasting constructions.) Thus, in general, there are strongly balanced, uniform RMDs with $n=\lambda_1 t^2$, $\lambda_1 \ge 1$, and $p=2\lambda_2 t$, $\lambda_2 \ge 1$ for all t. The design (a) in Table 2 is the strongly balanced, uniform RMD for t=2, p=4, n=4 from the construction of Cheng and Wu (1980). The designs (b) and (c) show strongly balanced, uniform RMD obtained from (a) by horizontal and vertical pasting respectively.

(a)

1	1	2	2
1	2	1	2
2	2	1	1
2	1	2	1

(b)

1	1	1	1	2	2	2	2
1	1	2	2	1	1	2	2
2	2	2	2	1	1	1	1
2	2	1	1	2	2	1	1

(c)

1	1	2	2
1	2	1	2
2	2	1	1
2	1	2	1
1	1	2	2
1	2	1	2
2	2	1	1
2	1	2	1

Table 2. *Examples of horizontal and vertical pasting.*

Sen and Mukerjee (1987) have shown how to construct a strongly balanced, uniform RMD for $n=t^2$ and $p=3t$. As their construction uses two mutually orthogonal Latin squares (MOLS) of size t, it can only be used when there are at least 3 treatments (and $t\neq 6$).

We are interested in the total number of strongly balanced, uniform RMDs and ways of constructing all these designs for small values of t and p for varying n. In the remainder of this paper, we consider the construction of non-isomorphic, strongly balanced, uniform RMDs for the cases t=2,p=4; t=2,p=6; t=2,p even,p>6 and t=3,p=6.

2. The Case t=2 and p=4 (=2t)

In this case, the necessary conditions for the existence of strongly balanced, uniform RMDs are $2|4$, $2|n$ and $4|3n$. Thus, $n=4s$, $s \geq 1$. Since the designs are uniform on units (or columns), each column of the array must contain two 1's and two 2's. Hence each experimental unit must receive one of six possible sequences. These are listed in Table 3.

Sequence		S_1	S_2	S_3	S_4	S_5	S_6
Period	1	1	1	1	2	2	2
	2	1	2	2	1	1	2
	3	2	1	2	1	2	1
	4	2	2	1	2	1	1

Table 3. *All sequences of length 4 containing two 1's and two 2's.*

For each sequence we have also recorded, in Table 4, the number of times the ordered pairs of treatments (1,1), (1,2), (2,1) and (2,2), appear on adjacent periods.

Sequence	S_1	S_2	S_3	S_4	S_5	S_6
$(1,1)^T$	1	0	0	1	0	1
$(1,2)^T$	1	2	1	1	1	0
$(2,1)^T$	0	1	1	1	2	1
$(2,2)^T$	1	0	1	0	0	1

Table 4. *The number of times the ordered pairs appear in each sequence.*

We let x_i, $i=1, 2,\ldots,6$, be the number of units receiving treatment sequence S_i in the design. Then, counting experimental units and using the fact that the design is both strongly balanced and uniform in rows (periods), we get the following equations.

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= n = 4s,\\
x_1 + x_4 + x_6 &= (p-1)n/t^2 = 3n/4 = 3s,\\
x_1 + 2x_2 + x_3 + x_4 + x_5 &= 3s,\\
x_2 + x_3 + x_4 + 2x_5 + x_6 &= 3s,\\
x_1 + x_3 + x_6 &= 3s,\\
x_1 + x_2 + x_3 &= n/t = n/2 = 2s,\\
x_1 + x_4 + x_5 &= 2s,\\
x_2 + x_4 + x_6 &= 2s,\\
x_3 + x_5 + x_6 &= 2s.
\end{aligned}$$

Solving, we get

$$\begin{aligned} x_1 &= x_6 = x_2 + s, \\ x_2 &= x_5, \\ x_3 &= x_4 = s - 2x_2, \\ 0 &\leq x_2 \leq \left[\tfrac{s}{2}\right], \quad s=1,2,3,\ldots, \end{aligned}$$

where $\left[\frac{s}{2}\right]$ is the largest integer less than or equal to s/2.

We summarise these results in the following theorem.

Theorem 1

When t=2 and p=4, all strongly balanced, uniform RMDs have n=4s units, s=1,2,... . There are $\left[\frac{s}{2}\right]$ + 1 non-isomorphic designs with 4s units and these designs have $a+s$ sequences of type S_1, and of type S_6, a sequences of type S_2, and of type S_5, and $s-2a$ sequences of type S_3, and of type S_4, where a =0,1,2,..., $\left[\frac{s}{2}\right]$. All the designs have (1,2) as an automorphism. □

In fact, all the designs are obtained by taking appropriate combinations of the design $(x_1,x_2,x_3) = (1,0,1)$ when n=4 and the design $(x_1,x_2,x_3) = (3,1,0)$ when n=8. This can be seen from Table 5 where all strongly balanced, uniform RMDs for n=4,8,12,16 and 20 are given. The designs constructed by Cheng and Wu (1980) correpsond to the case a =0 of the theorem.

n=4	1 1 2 2
	1 2 1 2
	2 2 1 1
	2 1 2 1

n=8	1 1 1 1 2 2 2 2
	1 1 2 2 1 1 2 2
	2 2 2 2 1 1 1 1
	2 2 1 1 2 2 1 1
	1 1 1 1 2 2 2 2
	1 1 1 2 1 2 2 2
	2 2 2 1 2 1 1 1
	2 2 2 2 1 1 1 1

n=12	1 1 1 1 1 1 2 2 2 2 2 2
	1 1 1 2 2 2 1 1 1 2 2 2
	2 2 2 2 2 2 1 1 1 1 1 1
	2 2 2 1 1 1 2 2 2 1 1 1
	1 1 1 1 1 1 2 2 2 2 2 2
	1 1 1 1 2 2 1 1 2 2 2 2
	2 2 2 2 1 2 1 2 1 1 1 1
	2 2 2 2 2 1 2 1 1 1 1 1

n=16	1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2
	1 1 1 1 2 2 2 2 1 1 1 1 2 2 2 2
	2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1
	2 2 2 2 1 1 1 1 2 2 2 2 1 1 1 1
	1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2
	1 1 1 1 1 2 2 2 1 1 1 2 2 2 2 2
	2 2 2 2 2 1 2 2 1 1 2 1 1 1 1 1
	2 2 2 2 2 2 1 1 2 2 1 1 1 1 1 1
	1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2
	1 1 1 1 1 1 2 2 1 1 2 2 2 2 2 2
	2 2 2 2 2 2 1 1 2 2 1 1 1 1 1 1
	2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1

n=20	1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2
	1 1 1 1 1 2 2 2 2 2 1 1 1 1 1 2 2 2 2 2
	2 2 2 2 2 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1
	2 2 2 2 2 1 1 1 1 1 2 2 2 2 2 1 1 1 1 1
	1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2
	1 1 1 1 1 1 2 2 2 2 1 1 1 1 2 2 2 2 2 2
	2 2 2 2 2 2 1 2 2 2 1 1 1 2 1 1 1 1 1 1
	2 2 2 2 2 2 2 1 1 1 2 2 2 1 1 1 1 1 1 1
	1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 2 2
	1 1 1 1 1 1 1 2 2 2 1 1 1 2 2 2 2 2 2 2
	2 2 2 2 2 2 2 1 1 2 1 2 2 1 1 1 1 1 1 1
	2 2 2 2 2 2 2 2 2 1 2 1 1 1 1 1 1 1 1 1

Table 5. *All strongly balanced, uniform RMDs for t=2, p=4 and n=4,8,12,16 and 20.*

3. The case t=2 and p=6 (=3t)

Here the necessary conditions for the existence of strongly balanced, uniform RMDs are $2|6$, $2|n$ and $4|5n$, so again $n=4s$, $s \geq 1$. There are now twenty possible sequences, each containing three 1's and three 2's. They are listed in Table 6.

Sequence	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}	S_{19}	S_{20}
Period 1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2
2	1	1	1	1	2	2	2	2	2	2	1	1	1	1	1	1	2	2	2	2
3	1	2	2	2	1	1	2	1	2	2	1	1	2	1	2	2	1	1	1	2
4	2	1	2	2	1	2	1	2	1	2	1	2	1	2	1	2	1	1	2	1
5	2	2	1	2	2	1	1	2	2	1	2	1	1	2	2	1	1	2	1	1
6	2	2	2	1	2	2	2	1	1	1	2	2	2	1	1	1	2	1	1	1

Table 6. *All sequences of length 6 containing three 1's and three 2's.*

For each sequence, the number of times that the ordered pairs of treatments (1,1), (1,2), (2,1) and (2,2) appear on adjacent periods, are recorded in Table 7.

Sequence	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}	S_{19}	S_{20}
$(1,1)^T$	2	1	1	1	1	0	1	0	0	1	2	1	1	1	0	1	2	1	1	2
$(1,2)^T$	1	2	2	1	2	3	2	2	2	1	1	2	2	1	2	1	1	1	1	0
$(2,1)^T$	0	1	1	1	1	2	1	2	2	1	1	2	2	2	3	2	1	2	2	1
$(2,2)^T$	2	1	1	2	1	0	1	1	1	2	1	0	0	1	0	1	1	1	1	2

Table 7. *The number of times the ordered pairs (1,1), (1,2), (2,1), (2,2) appear in each sequence.*

We let x_i, $i=1,2,..,20$, be the number of units receiving treatment sequence S_i in the design. Then we can obtain the following equations in a similar way to those of the previous case , using the fact that the design is both strongly balanced and uniform on rows (periods).

$$2x_1+x_2+x_3+x_4+x_5+x_7+x_{10}+2x_{11}+x_{12}+x_{13}+x_{14}+x_{16}+2x_{17}+x_{18}+x_{19}+2x_{20} = \frac{(p-1)n}{t^2} = 5s,$$
$$x_1+2x_2+2x_3+x_4+2x_5+3x_6+2x_7+2x_8+2x_9+x_{10}+x_{11}+2x_{12}+2x_{13}+x_{14}+2x_{15}+x_{16}+x_{17}+x_{18}+x_{19} = 5s,$$
$$x_2+x_3+x_4+x_5+2x_6+x_7+2x_8+2x_9+x_{10}+x_{11}+2x_{12}+2x_{13}+2x_{14}+3x_{15}+2x_{16}+x_{17}+2x_{18}+2x_{19}+x_{20} = 5s,$$
$$2x_1+x_2+x_3+2x_4+x_5+x_7+x_8+x_9+2x_{10}+x_{11}+x_{14}+x_{16}+x_{17}+x_{18}+x_{19}+2x_{20} = 5s,$$

$$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10} = \frac{n}{t} = 2s,$$
$$x_1+x_2+x_3+x_4+x_{11}+x_{12}+x_{13}+x_{14}+x_{15}+x_{16} = 2s,$$
$$x_1+x_5+x_6+x_8+x_{11}+x_{12}+x_{14}+x_{17}+x_{18}+x_{19} = 2s,$$
$$x_2+x_5+x_7+x_9+x_{11}+x_{13}+x_{15}+x_{17}+x_{18}+x_{20} = 2s,$$
$$x_3+x_6+x_7+x_{10}+x_{12}+x_{13}+x_{16}+x_{17}+x_{19}+x_{20} = 2s,$$
$$x_4+x_8+x_9+x_{10}+x_{14}+x_{15}+x_{16}+x_{18}\ x_{19}+x_{20} = 2s.$$

Solving these equations, we find that

$$x_1 = x_7+x_9+x_{10}+x_{13}+x_{15}+x_{16}+x_{17}+x_{18}+x_{19}+2x_{20}-2s,$$
$$x_2 = x_7+2x_{10}-x_{15}+x_{16}+2x_{17}+x_{18}+2x_{19}+3x_{20}-3s,$$
$$x_3 = -2x_7+x_8-2x_{10}-x_{13}-x_{14}+x_{15}-x_{16}-2x_{17}-x_{19}-3x_{20}+3s,$$
$$x_4 = -x_8-x_9-x_{10}-x_{14}-x_{15}-x_{16}-x_{18}-x_{19}-x_{20}+2s,$$
$$x_5 = -2x_7-x_9-2x_{10}+x_{12}+x_{14}+x_{15}-2x_{17}-x_{18}-x_{19}-3x_{20}+3s,$$
$$x_6 = x_7-x_8+x_{10}-x_{12}-x_{14}-x_{15}+x_{17}+2x_{20}-s,$$
$$x_{11} = -x_{12}-x_{13}-x_{14}-x_{15}-x_{16}-x_{17}-x_{18}-x_{19}-x_{20}+2s,$$

$$0 \le x_7, x_8, x_9, x_{10}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20} \le 2s,\ s = 1,2,3,\ldots .$$

In the previous section, we could find all strongly balanced, uniform RMDs from the equivalent equations. Here we cannot find all solutions to the above equations very easily. When s is any integer value then the set of solutions to the above equations are a module which is finitely generated. The RMDs correspond to the positive elements of the module (De Launey (1989)). However, the basis of the module may only be expressible as linear combinations of the original x_i's. Hence this observation does not appear to make the task of finding the designs any easier.

It is no longer true that all solutions have (1,2) as an automorphism. Those that do are called *symmetric* designs. Otherwise, we say that the design is *non-symmetric*. In a symmetric design $x_1=x_{20}$, $x_2=x_{19},\ldots,x_{10}=x_{11}$.

All non-isomorphic strongly balanced, uniform RMDs with t=2, p=6 and n=4, are given in Table 8. We see that there are 15 designs, of which the first 10 are symmetric.

1 1 2 2	1 1 2 2	1 1 2 2	1 1 2 2	1 1 2 2	
1 2 1 2	1 2 1 2	1 1 2 2	1 2 1 2	1 1 2 2	
1 1 2 2	1 2 1 2	2 2 1 1	2 2 1 1	2 2 1 1	
2 2 1 1	2 1 2 1	1 2 1 2	1 2 1 2	2 2 1 1	
2 2 1 1	2 2 1 1	2 2 1 1	2 1 2 1	1 2 1 2	
2 1 2 1	2 1 2 1	2 1 2 1	2 1 2 1	2 1 2 1	*symmetric*
1 1 2 2	1 1 2 2	1 1 2 2	1 1 2 2	1 1 2 2	
1 2 1 2	1 2 1 2	1 2 1 2	2 2 1 1	2 2 1 1	
2 2 1 1	2 1 2 1	2 2 1 1	1 2 1 2	2 2 1 1	
2 2 1 1	2 1 2 1	2 1 2 1	1 2 1 2	1 2 1 2	
1 1 2 2	2 2 1 1	2 1 2 1	2 1 2 1	1 1 2 2	
2 1 2 1	1 2 1 2	1 2 1 2	2 1 2 1	2 1 2 1	

1 1 2 2	1 1 2 2	1 1 2 2	1 1 2 2	1 1 2 2	
1 2 1 2	1 2 1 2	1 2 1 2	1 2 1 2	1 2 1 2	
1 2 2 1	1 2 2 1	1 2 2 1	1 2 2 1	2 2 1 1	*non-symmetric*
2 1 2 1	2 1 2 1	2 2 1 1	2 2 1 1	1 2 2 1	
2 1 1 2	2 2 1 1	2 1 1 2	2 1 2 1	2 1 2 1	
2 2 1 1	2 1 1 2	2 1 2 1	2 1 1 2	2 1 1 2	

Table 8. *Strongly balanced, uniform RMD's for t=2, p=6, n=4.*

The number of non-isomorphic (symmetric and non-symmetric), strongly balanced, uniform RMDs with t=2, p=6 and n=4,8 and 12, are given in Table 9.

n	4	8	12
symmetric	10	84	388
non-symmetric	5	130	1636
Total	15	214	2024

Table 9. *The number of non-isomorphic strongly balanced, uniform RMDs for t=2 and p=6.*

When t=2 and p=4, design with n=4s, s≥3, are obtained by horizontally pasting an appropriate number of designs with n=4 and n=8, and they can be obtained in no other way. This is no longer the case when t=2 and p=6. Table 10 shows the number of designs with n=8 which can be obtained by horizontally pasting designs with n=4. (We used all 20 designs with n=4 - the 10 symmetric and 5 non-symmetric designs from Table 8, and the 5 designs obtained from the non-symmetric designs by applying the permutation

(12).) Table 10 also gives the number of designs with n=12 which can be obtained by horizontally pasting a design with n=4 and one with n=8 (84 symmetric, 130 non-symmetric and 130 non-symmetric permuted (1,2)).

n	8	12
symmetric	51	38
non-symmetric	75	1494
Total	126	1874

Table 10. *The number of non-isomorphic, strongly balanced, uniform RMDs possible using pasting for n=8 and 12.*

Non-symmetric designs permuted by (1 2) are included for pasting as they may lead to designs which cannot be obtained otherwise. Table 11 gives a design for n=8 which is obtained by horizontally pasting a non-symmetric design for n=4 with 1 and 2 interchanged and a n=4 design from Table 8. This design cannot be obtained by pasting any two designs in Table 8. For n=8 there are 12 such strongly balanced, uniform RMDs, two of which are symmetric.

1	1	2	2	:	2	2	1	1
1	2	1	2	:	2	1	2	1
1	2	2	1	:	2	1	1	2
2	1	2	1	:	1	2	1	2
2	1	1	2	:	1	2	2	1
2	2	1	1	:	1	1	2	2

Table 11. *A strongly balanced, uniform RMD for t=2, p=6 and n=8 pasted from a non-symmetric design with the non-symmetric design permuted by (12).*

Pasting doesn't lead us to all designs: For n=8 there are 88 'new' designs which cannot be obtained from n=4 designs and for n=12, there are 150 'new' designs.

4. The Case t=2 and p>6 (even)

In this section we describe another recursive construction for strongly balanced uniform RMDs. Using Theorem 2 it is possible to construct such designs for t=2, n=4s and p=8,10,12,... .

Theorem 2

Let D_1 be a strongly balanced, uniform RMD with t=2, $p=p_1$ and n units, and let D_2 be a strongly balanced, uniform RMD with t=2, $p=p_2$ and n units. Then there is a strongly balanced, uniform RMD with t=2, $p=p_1+p_2$ and n units.

Proof

We can permute the columns of D_1 so that the first n/2 columns have a 1 in the final row (and hence the remaining columns have a 2 in the final row). We can permute the columns of D_2 so that the first n/4 columns begin with 1, the next n/4 columns begin with 2, the next n/4 columns begin with 1 and the final n/4 columns begin with 2. The required design is

$$D_3 = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

Clearly D_3 has t=2, $p=p_1+p_2$ and there are n units. D_3 is uniform in rows and columns because D_1 and D_2 were. What are the values of m_{ij} in D_3? From D_1 and D_2 and the method of construction we have that

$$m_{ij} = \frac{n(p_1-1)}{4} + \frac{n(p_2-1)}{4} + \frac{n}{4} = \frac{n(p_1-1+p_2-1+1)}{4},$$

as required. □

The designs in Table 12 illustrate this construction. The first is a t=2, p=8, n=8 design obtained from two (different) designs with t=2, p=4, n=8. The second is a design with t=2, p=10, n=4 obtained from designs with t=2, p=6, n=4 and t=2, p=4, n=4. The third design has t=2, p=12, n=4 and is obtained from two (different) designs with t=2, p=6, n=4.

(a)

2	2	2	2	1	1	1	1
1	2	2	2	1	1	1	2
2	1	1	1	2	2	2	1
1	1	1	1	2	2	2	2
1	1	2	2	1	1	2	2
1	1	1	1	2	2	2	2
2	2	1	1	2	2	1	1
2	2	2	2	1	1	1	1

(b)

2	2	1	1
1	2	1	2
2	1	1	2
2	1	2	1
1	2	2	1
1	1	2	2
1	2	1	2
1	1	2	2
2	1	2	1
2	2	1	1

(c)

1	2	1	2
2	1	2	1
2	1	2	1
2	2	1	1
1	2	1	2
1	1	2	2
1	2	1	2
1	1	2	2
2	1	2	1
1	2	2	1
2	2	1	1
2	1	1	2

Table 12. *Strongly balanced, uniform RMDs obtained by vertical pasting.*

5. The Case t=3 and p=6 (=2t)

Once we have more than two treatments, the situation rapidly becomes much more complicated. We illustrate some of these difficulties by considering the case t=3 and p=6.

There are now 90 sequences of length 6 which contain two 1's, two 2's and two 3's. These can be grouped into 15 sets of 6 sequences each, where sequences in a set can be obtained from each other by applying a permutation of 1,2 and 3 (that is, an element of S_3).

The 90 sequences, grouped into the 15 sets of 6, together with a label for each sequence, appear in Table 13.

	1 1 2 2 3 3	s_{11}	1 1 2 3 2 3	s_{21}	1 1 2 3 3 2	s_{31}
(12)	2 2 1 1 3 3	s_{12}	2 2 1 3 1 3	s_{22}	2 2 1 3 3 1	s_{32}
(13)	3 3 2 2 1 1	s_{13}	3 3 2 1 2 1	s_{23}	3 3 2 1 1 2	s_{33}
(23)	1 1 3 3 2 2	s_{14}	1 1 3 2 3 2	s_{24}	1 1 3 2 2 3	s_{34}
(123)	2 2 3 3 1 1	s_{15}	2 2 3 1 3 1	s_{25}	2 2 3 1 1 3	s_{35}
(132)	3 3 1 1 2 2	s_{16}	3 3 1 2 1 2	s_{26}	3 3 1 2 2 1	s_{36}
	1 2 1 2 3 3	s_{41}	1 2 1 3 2 3	s_{51}	1 2 1 3 3 2	s_{61}
(12)	2 1 2 1 3 3	s_{42}	2 1 2 3 1 3	s_{52}	2 1 2 3 3 1	s_{62}
(13)	3 2 3 2 1 1	s_{43}	3 2 3 1 2 1	s_{53}	3 2 3 1 1 2	s_{63}
(23)	1 3 1 3 2 2	s_{44}	1 3 1 2 3 2	s_{54}	1 3 1 2 2 3	s_{64}
(123)	2 3 2 3 1 1	s_{45}	2 3 2 1 3 1	s_{55}	2 3 2 1 1 3	s_{65}
(132)	3 1 3 1 2 2	s_{46}	3 1 3 2 1 2	s_{56}	3 1 3 2 2 1	s_{66}
	1 2 2 1 3 3	s_{71}	1 2 3 1 2 3	s_{81}	1 2 3 1 3 2	s_{91}
(12)	2 1 1 2 3 3	s_{72}	2 1 3 2 1 3	s_{82}	2 1 3 2 3 1	s_{92}
(13)	3 2 2 3 1 1	s_{73}	3 2 1 3 2 1	s_{83}	3 2 1 3 1 2	s_{93}
(23)	1 3 3 1 2 2	s_{74}	1 3 2 1 3 2	s_{84}	1 3 2 1 2 3	s_{94}
(123)	2 3 3 2 1 1	s_{75}	2 3 1 2 3 1	s_{85}	2 3 1 2 1 3	s_{95}
(132)	3 1 1 3 2 2	s_{76}	3 1 2 3 1 2	s_{86}	3 1 2 3 2 1	s_{96}
	1 2 2 3 1 3	$s_{10,1}$	1 2 3 2 1 3	$s_{11,1}$	1 3 2 2 1 3	$s_{12,1}$
(12)	2 1 1 3 2 3	$s_{10,2}$	2 1 3 1 2 3	$s_{11,2}$	2 3 1 1 2 3	$s_{12,2}$
(13)	3 2 2 1 3 1	$s_{10,3}$	3 2 1 2 3 1	$s_{11,3}$	3 1 2 2 3 1	$s_{12,3}$
(23)	1 3 3 2 1 2	$s_{10,4}$	1 3 2 3 1 2	$s_{11,4}$	1 2 3 3 1 2	$s_{12,4}$
(123)	2 3 3 1 2 1	$s_{10,5}$	2 3 1 3 2 1	$s_{11,5}$	2 1 3 3 2 1	$s_{12,5}$
(132)	3 1 1 2 3 2	$s_{10,6}$	3 1 2 1 3 2	$s_{11,6}$	3 2 1 1 3 2	$s_{12,6}$
	1 2 2 3 3 1	$s_{13,1}$	1 2 3 2 3 1	$s_{14,1}$	1 2 3 3 2 1	$s_{15,1}$
(12)	2 1 1 3 3 2	$s_{13,2}$	2 1 3 1 3 2	$s_{14,2}$	2 1 3 3 1 2	$s_{15,2}$
(13)	3 2 2 1 1 3	$s_{13,3}$	3 2 1 2 1 3	$s_{14,3}$	3 2 1 1 2 3	$s_{15,3}$
(23)	1 3 3 2 2 1	$s_{13,4}$	1 3 2 3 2 1	$s_{14,4}$	1 3 2 2 3 1	$s_{15,4}$
(123)	2 3 3 1 1 2	$s_{13,5}$	2 3 1 3 1 2	$s_{14,5}$	2 3 1 1 3 2	$s_{15,5}$
(132)	3 1 1 2 2 3	$s_{13,6}$	3 1 2 1 2 3	$s_{14,6}$	3 1 2 2 1 3	$s_{15,6}$

Table 13. *90 possible sequences for $t=3$ and $p=6$.*

Suppose there are x_{ij} units receiving treatment sequence S_{ij} in the final design. Then uniformity in rows gives us 3x6 = 18 equations and the strongly balanced property gives us a further 9 equations. However, the equations are not independent (for instance as there can only be 1's, 2's and 3's in each row, once the number of 1's and 2's are known, the number of 3's is also). The 27 equations in fact have rank 15 and involve 90 unknown x_{ij}.

We simplify the problem further by finding only those designs for which all the elements of S_3 are an automorphism. Thus n=18s and there are two independent equations that the 15 unknowns must satisfy.

Let x_i be the number of sequences of type S_{i1} in the final design.

Then

$$\sum_{i=1}^{15} x_i = n/6 = 3s \tag{1}$$

$$6x_1 + 2x_2 + 4x_3 + 2x_4 + 2x_6 + 4x_7 + 2x_{10} + 2x_{12} + 4x_{13} + 2x_{15} = \frac{5n}{9} = 10s \tag{2}$$

$$2x_1+4x_2+3x_3+4x_4+5x_5+4x_6+3x_7+5x_8+5x_9+4x_{10}+5x_{11}+4x_{12}+3x_{13}+5x_{14}+4x_{15} = 10s \tag{3}$$

In attempting to find solutions it appears to be easiest to work with the original equations.

Theorem 3

There are 72 non-isomorphic, strongly balanced, uniform RMDs with t=3, p=6, n=18 and with S_3 as an automorphism group.

Proof

Any such design must satisfy the equations (1), (2) and (3) with s=1. Thus $0 \le x_i \le 3$. But if any $x_i = 3$, then either equation (2) or (3) is contradicted. If x_5, x_8, x_9, x_{11} or $x_{14} = 2$, then (1) and (3) can not hold simultaneously. If $x_1 = 2$, then equation (2) is false.

If $x_1 = 1$, then either one of x_3, x_7 and x_{13} is 1 and one of x_5, x_8, x_9, x_{11} and x_{14} is 1, or two of x_2, x_4, x_6, x_{10}, x_{12}, x_{15} is 1, or one of x_2, x_4, x_6, x_{10}, x_{12} and x_{15} is 2. This gives 3 x 5 + 15 + 6 = 36 designs.

If $x_1 = 0$, then either one of x_3, x_7 and x_{13} is 2 and one of x_2, x_4, x_6, x_{10}, x_{12} and x_{15} is 1 or two of x_3, x_7 and x_{13} are 1 and one of x_2, x_4 x_6, x_{10}, x_{12} and x_{15} is 1. This gives 3 x 6 + 3 x 6 = 36 designs. The result follows. □

Similar counting shows that there are 1677 such designs when n=36 (s=2).

5. Summary

In this paper we have produced constructions for all strongly balanced, uniform RMDs for t=2, p=4 and n=4s. All strongly balanced, uniform RMDs for t=2, p=6 and n=4 have been given from which we can horizontally paste to produce some t=2, p=6 and n=4s designs. Using t=2, p=4, 6 and n=4s we can construct t=2, p>6 (even) and n=4s strongly balanced, uniform RMDs using vertical pasting. For t=3, p=6 and n=18, 36 we have counted the number of strongly balanced, uniform RMDs that have elements of S_3 as an automorphism.

Acknowledgements

The authors thank the Biometry Section, Waite Agricultural Research Institute, The University of Adelaide for support with this work, and Dr. S.R. Eckert for useful comments and advice on programming.

References

Cheng, C.S. and Wu, C.F. (1980). Balanced repeated measurements designs. *Annals of Statistics* **8**, 1272-1283.

De Launey, W. (1989). Personal communication.

Kunert, J. (1984). Optimality of balanced uniform repeated measurements designs. *Annals of Statistics* **12**, 1006-1017.

Sen, M. and Mukerjee, R. (1987). Optimal repeated measurements designs under interation. *Journal of Statistical Planing and Inference* **17**, 81-91.

Street, D.J. (1988). Combinatorial problems in repeated measurements designs. To appear in *Annals of Discrete Mathematics* .